# SCHOFIELD INDUCTION FOR SHEAVES ON WEIGHTED PROJECTIVE LINES

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Abstract. We show that each exceptional vector bundle on a weighted projective line in the sense of Geigle and Lenzing can be obtained by Schofield induction from exceptional sheaves of rank one and zero. This relates to results of Ringel concerning modules over finite dimensional $k$-algebras over an arbitrary field.

## 1. Introduction

Schofield induction [13] was applied by Ringel [11] in hereditary length categories. In [12] Ringel used this construction to show that each exceptional representation for a finite quiver without oriented cycles can be exhibited by matrices having as coefficients only 0 and 1. Recall that an object $M$ in a hereditary category is called exceptional if $\mathrm{Ext}^1(M,M)=0$ and $\mathrm{End}(M)$ is a skew field.

In this paper we study Schofield induction in the category of coherent sheaves over weighted projective lines in the sense of Geigle and Lenzing [2]. The main result is the following theorem.

**Theorem 1.** *Let $M$ be an exceptional vector bundle of rank greater than one on a weighted projective line $\mathbb{X}$ over an algebraically closed field. Then there are exceptional coherent sheaves $X, Y$ with the properties*

($i$) $\mathrm{Hom}_{\mathbb{X}}(X,Y)=\mathrm{Hom}_{\mathbb{X}}(Y,X)=\mathrm{Ext}^1_{\mathbb{X}}(Y,X)=0$ *and there is a non-split exact sequence*

$$\eta : 0 \to Y^v \to M \to X^u \to 0$$

*where $[u\,v]$ is the dimension vector of an exceptional representation of $\Theta(n)$ and $n=\dim \mathrm{Ext}^1_{\mathbb{X}}(X,Y)$.*

($ii$) $\mathrm{rk}Y < \mathrm{rk}M$ *and* $\mathrm{rk}X < \mathrm{rk}M$.

Here $\Theta(n)$ denotes the quiver

$$1 \;\underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}}\; 2 \quad (n \quad \text{arrows}).$$

and rk denotes the rank of a coherent sheaf. If $\eta$ is as in ($i$) then the condition $\mathrm{rk}X < \mathrm{rk}M$ is clearly satisfied since the rank function is additive and $Y$ as a subobject of a vector bundle has positive rank. We emphasize, however, that it is important to have the other condition $\mathrm{rk}Y < \mathrm{rk}M$, because otherwise we cannot apply induction. In order to show this we use a result of Geigle and Lenzing concerning perpendicular categories. Note that in the case of modules over a finite dimensional path algebra the role of the rank is played by the dimension and then both conditions of ($ii$) are automatically satisfied. Moreover we introduce the

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concept of an *adapted tilting bundle* in order to have similar conditions as in [11, Lemma 3.3]. Finally, we show that in our situation the last term of the exact sequence $\eta$ can be chosen as a vector bundle or a simple sheaf of finite length.

The exceptional vector bundles of rank one are well known. They are described in [9]. On the other hand also the exceptional sheaves of rank zero are known (see [4, Proposition 4.2, Proposition 4.4]). In that paper the case of three weights is considered, however the results easily generalize to the general situation. Then applying the theorem successively we can obtain all exceptional vector bundles on $\mathbb{X}$ from exceptional sheaves of rank one or zero. In [12] Ringel used Schofield induction to show that each exceptional module over a path algebra of a quiver can be represented by matrices having as coefficients 0 and 1. We are going to prove a similar result for exceptional modules over canonical algebra in the sense of [10]. For this one should use that the derived category of modules over a canonical algebra is equivalent to the derived category of coherent sheaves over the corresponding weighted projective line. In contrast to the situation of path algebras also the parameters in the definition of the algebra will play a role. We will study this problem in a forthcoming paper.

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## 2. Notations

We recall the concept of weighted projective lines and refer the reader for details to [2].

**2.1.** Throughout this paper $k$ denotes an algebraically closed field. Let $\mathbf{L}(\mathbf{p})$ be the rank one abelian group on generators $\vec{x}_1$, $\vec{x}_2,\ldots,\vec{x}_t$ with relations $p_1\vec{x}_1 = p_2\vec{x}_2 = \cdots = p_t\vec{x}_t =: \vec{c}$. Further for a sequence of integers $\mathbf{p} = (p_1,\ldots,p_t)$ consider the algebra $k[X_1,\ldots,X_t]$ with the $\mathbf{L}(\mathbf{p})$ grading defined by $\deg X_i = \vec{x}_i$. Moreover for a sequence of parameters $\lambda = (\lambda_3,\ldots,\lambda_t)$ we denote by $S = S(\mathbf{p},\lambda) = k[X_1, X_2,\ldots,X_t]/I(\mathbf{p},\lambda)$ where $I(\mathbf{p},\lambda)$ is the ideal generated by the elements $X_i^{p_i} - X_2^{p_2} + \lambda_i X_1^{p_1}, i = 3, ..., t$. Because $I(\mathbf{p},\lambda)$ is a homogeneous ideal the algebra $S = S(\mathbf{p},\lambda)$ is $\mathbf{L}(\mathbf{p})$-graded. The weighted projective line $\mathbb{X} = \mathbb{X}(\mathbf{p},\lambda)$ is by definition the projective spectrum of the $\mathbf{L}(\mathbf{p})$-graded algebra $S(\mathbf{p},\lambda)$. We denote by $\mathrm{coh}(\mathbb{X})$ the category of $\mathbf{L}(\mathbf{p})$ graded coherent sheaves on $\mathbb{X}$ and by $\mathrm{vect}(\mathbb{X})$ (respectively $\mathrm{coh}_0(\mathbb{X})$) the full subcategory of vector bundles (respectively finite length sheaves). Note that there are no non-zero morphisms from objects $\mathrm{coh}_0(\mathbb{X})$ to objects from $\mathrm{vect}(\mathbb{X})$. The category $\mathrm{coh}(\mathbb{X})$ is hereditary and admits Auslander-Reiten sequences. The Auslander-Reiten translation $\tau_{\mathbb{X}}$ is given by degree shift with the dualizing element $\vec{\omega} = (t-2)\vec{c} - \sum_{i=1}^{t}\vec{x}_i$ and gives rise to Serre duality $\mathrm{DExt}^1_{\mathbb{X}}(F,G) \simeq \mathrm{Hom}_{\mathbb{X}}(G,\tau_{\mathbb{X}}F)$.

**2.2.** We will need the concept of tilting sheaves and exceptional sequences in $\mathrm{coh}(\mathbb{X})$. For details concerning this subject we refer to [2], [5].

**Definition 2.2.** An object $T \in \mathrm{coh}(\mathbb{X})$ is called a *tilting sheaf* if the following two conditions are satisfied

(i) $\mathrm{Ext}^1_{\mathbb{X}}(T,T) = 0$;

(ii) If $X \in \mathrm{coh}(\mathbb{X})$ satisfies $\mathrm{Hom}_{\mathbb{X}}(T,X) = 0 = \mathrm{Ext}^1_{\mathbb{X}}(T,X)$ then $X = 0$.

A tilting sheaf for which every indecomposable direct summand belongs to $\mathrm{vect}(\mathbb{X})$ is shortly called a *tilting bundle*.

In [2] it was shown that $T = \bigoplus_{0\leq\vec{x}\leq\vec{c}} \mathcal{O}(\vec{x})$ is a tilting bundle, which is called furtheron the *canonical tilting bundle*. Here $\mathbf{L}(\mathbf{p})$ is considered as an ordered group where $\sum_{i=1}^{t} \mathbb{N}\vec{x}_i$ is its set of non-negative elements.

A sequence of exceptional objects in $\mathrm{coh}(\mathbb{X})$ of the form $(M_1, \dots, M_r)$ is called an *exceptional sequence* if for all $i > j$ we have $\mathrm{Hom}_{\mathbb{X}}(M_i, M_j) = 0 = \mathrm{Ext}^1_{\mathbb{X}}(M_i, M_j)$ [8]. We denote by $s$ the rank of the Grothendieck group $K_0(\mathbb{X}) := K_0(\mathrm{coh}(\mathbb{X}))$. An exceptional sequence in $\mathrm{coh}(\mathbb{X})$ of length $s$ is called a *complete exceptional sequence* and an exceptional sequence in $\mathrm{coh}(\mathbb{X})$ of length 2 is called an *exceptional pair*. It is well known that the indecomposable direct summands of a multiplicity-free tilting sheaf in $\mathrm{coh}(\mathbb{X})$ can be ordered in such a way that they form a complete exceptional sequence.

## 3. Exceptional vector bundles over wild weighted projective lines

**3.1.** Assume that $\mathbb{X}$ is a weighted projective line of wild representation type. Let $M \in \mathrm{vect}(\mathbb{X})$ be an exceptional vector bundle.

**Definition 3.1.** We call a tilting bundle $T$ *adapted* to $M$ if the following two conditions are satisfied

(i) $\mathrm{Ext}^1_{\mathbb{X}}(T, M) = 0$;

(ii) There is a monomorphism $T \hookrightarrow M^{\oplus t}$ for some natural number $t$.

We recall that there is a function $\delta : \mathbf{L}(\mathbf{p}) \to \mathbb{Z}$ defined on generators by the formula $\delta(\vec{x}) = \frac{p}{p_i}$ where $p = \mathrm{l.c.m.}(p_1, \dots, p_t)$. Moreover, for a non-zero coherent sheaf on $\mathbb{X}$ the slope $\mu(E)$ is defined as the quotient $\frac{\deg E}{\mathrm{rk} E}$ where deg denotes the degree which is a linear form on $K_0(\mathbb{X})$ defined by $\deg \mathcal{O}(\vec{x}) = \delta(\vec{x})$ for $\vec{x} \in \mathbf{L}(\mathbf{p})$. Finally, a non-zero vector bundle $F$ is called semistable if for each non-zero subbundle $F'$ we have $\mu(F') \leq \mu(F)$. For more details we refer to [2].

**Lemma 3.1.** *For each vector bundle $M$ there exists a tilting bundle adapted to $M$.*

**Proof.** Let $T = \bigoplus_{0\leq\vec{x}\leq\vec{c}} \mathcal{O}(\vec{x})$ be the canonical tilting bundle. Let $\vec{y} \in \mathbf{L}(\mathbf{p})$ be such that $\mu(\mathcal{O}(\vec{x}+\vec{y}+\vec{\omega})) < \mu(M^s) - (\mathrm{rk} M - 1)\delta(\vec{\omega})$ for all $\vec{x}$ with $0 < \vec{x} < \vec{c}$ where $M^s$ denotes the maximal semistable subbundle of $M$. Then we conclude from [7, Theorem 2.9] that $\mathrm{Hom}_{\mathbb{X}}(M, T(\vec{y}+\vec{\omega})) = 0$. Applying Serre duality it follows that $\mathrm{Ext}^1_{\mathbb{X}}(T(\vec{y}), M) = 0$.

Assume moreover that $\vec{y}$ satisfies $\mu(\mathcal{O}(\vec{x}+\vec{y})) < \mu(M) - p - \delta(\vec{\omega})$ for each $\vec{x}$ with $0 < \vec{x} < \vec{c}$. Then by [7, Theorem 2.7] we have that $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}+\vec{y}), M) \neq 0$. Since by [5, Lemma 10.3] each non-zero morphism from a line bundle to a vector bundle is a monomorphism we obtain a monomorphism $T(\vec{y}) \hookrightarrow M^{\oplus t}$ for some natural number $t$. Thus $T(\vec{y})$ is a tilting bundle which is adapted to $M$. □

**3.2.** Let $M$ be an exceptional vector bundle. We choose a tilting bundle $T$ adapted to $M$. We denote $\dim \mathrm{Ext}^1_{\mathbb{X}}(M, T) = m$ and consider the *universal extension*

$$0 \to T \to M' \to M^{\oplus m} \to 0.$$

By definition the sequence above is a universal extension if the connecting homomorphism in the long exact sequence obtained by applying the functor $\mathrm{Hom}_{\mathbb{X}}(T, -)$ is an isomorphism. Then $M' \oplus M$ is a tilting bundle. The proof for modules over finite dimensional algebras [1] carries over to our situation, note that for this the condition $\mathrm{Ext}^1_{\mathbb{X}}(T, M) = 0$ is necessary. The bundle $M'$ which in general is not multiplicity-free is called the *Bongartz complement* of $M$ with respect to $T$.

We have the following two analogs of results proved in [11, Lemma 3.3, Lemma 3.4].

**Lemma 3.2.** *Let $M \in \mathrm{vect}(\mathbb{X})$ be an exceptional vector bundle, $T$ a tilting bundle adapted to $M$ and $M'$ the Bongartz complement of $M$ with respect to $T$. Then $M'$ is cogenerated by $M$ and therefore $\mathrm{Hom}_{\mathbb{X}}(M, M') = 0$.*

**Proof.** Since $T$ is adapted to $M$ there is a monomorphism $\alpha : T \to M^{\oplus t}$ for some natural number $t$. Then the arguments of [11, Lemma 3.3] can be applied, replacing the module $\Lambda$ by $T$. □

**Proposition 3.3.** *Let $M \in \mathrm{vect}(\mathbb{X})$ be an exceptional vector bundle, $T$ a tilting bundle adapted to $M$ and let $N$ be indecomposable in $\mathrm{vect}(\mathbb{X})$. The following conditions are equivalent:*

*(i) $N$ is a direct summand of the Bongartz complement of $M$ with respect to $T$.*

*(ii) $(N, M)$ is an exceptional pair in $\mathrm{coh}(\mathbb{X})$ and $N$ is cogenerated by $M$.*

The proof is a reproduction of [11, Lemma 3.3].

**3.4.** We have the following immediate consequence of Proposition 3.3

**Corollary 1.** *Let $M \in \mathrm{vect}(\mathbb{X})$ be an exceptional vector bundle, $T_1, T_2$ two tilting bundles which are both adapted to $M$. Further let $M'_i$ be the Bongartz complement of $M$ with respect to $T_i$ for $i = 1, 2$. Then $M'_1$ and $M'_2$ have the same indecomposable direct summands (possibly with different multiplicities).* □

**3.5.** We now prove Theorem 1.

(i) If $\mathbb{X}$ is of tubular representation type the theorem was proved in [9]. In this case the numbers $u$ and $v$ can be chosen to be 1. In the domestic case the statement is easy and follows from the structure of the Auslander-Reiten components of the category of vector bundles on $\mathbb{X}$. Therefore we can assume that $\mathbb{X}$ is of wild representation type.

We choose a tilting bundle $T$ adapted to $M$. Let $N_1, \dots, N_{s-1}$ be pairwise non-isomorphic direct summands of the Bongartz complement $M'$ of $M$ with respect to $T$. We denote by $\mathcal{C}_i = \mathcal{C}(N_i, M)$ the closure of the full subcategory of $\mathrm{coh}(\mathbb{X})$, with objects $N_i, M$, under kernels, images, cokernels and extensions. Then for each $i$ the category $\mathcal{C}_i$ is an exact abelian subcategory of $\mathrm{coh}(\mathbb{X})$. Moreover, since $\mathrm{rk}M \geq 2$ by applying [6, Corollary 1] and [6, Theorem 1] we deduce that each $\mathcal{C}_i$ is equivalent to a module category of a finite dimensional $k$-algebra with two simple modules.

Thus $\mathcal{C}_i \simeq \mathrm{mod}(H_{n_i})$ for some generalized Kronecker algebra $H_{n_i} = k\Theta(n_i)$, where $n_i$ denotes the number of arrows. Denote for each $i$ by $Y_i$ (respectively $X_i$) the simple projective (respectively injective) module in $\mathcal{C}_i$.

We conclude from Proposition 3.3 that $(N_i, M)$ is an exceptional pair and that $N_i$ is cogenerated by $M$. It follows from [11, Lemma 3.2] that $M$ is not simple in $\mathcal{C}_i$. This implies that there is an exact sequence

$$\eta_i : 0 \to Y_i^{v_i} \to M \to X_i^{u_i} \to 0$$

for each $i = 1, \dots, s-1$, where $u_i, v_i \geq 1$.

(ii) Keeping the notations above we show that there is an $i$ such that $\mathrm{rk}Y_i < \mathrm{rk}M$. Assume contrary that $\mathrm{rk}Y_i = \mathrm{rk}M$ for all $i$. Then $v_i = 1$ and $\mathrm{rk}(X_i) = 0$. Now we have exact sequences

$$\eta_i : 0 \to Y_i \to M \to X_i^{u_i} \to 0.$$

Calculating dimension vectors we conclude that $M$ is injective in $\mathcal{C}_i$ and $u_i = n_i$. Therefore $\mathrm{Ext}^1_{\mathbb{X}}(X_i, M) = \mathrm{Ext}^1_{\mathcal{C}_i}(X_i, M) = 0$ and $\mathrm{Ext}^1_{\mathbb{X}}(M, X_i) = \mathrm{Ext}^1_{\mathcal{C}_i}(M, X_i) = 0$ for each $i$. Furthermore, applying the functor $\mathrm{Hom}_{\mathbb{X}}(X_j, -)$ to $\eta_i$, we see that there is an epimorphism $\mathrm{Ext}^1_{\mathbb{X}}(X_j, M) \to \mathrm{Ext}^1_{\mathbb{X}}(X_j, X_i^{u_i})$. Since $\mathrm{Ext}^1_{\mathbb{X}}(X_j, M) = \mathrm{Ext}^1_{\mathcal{C}_j}(X_j, M) = 0$ it follows that $\mathrm{Ext}^1_{\mathbb{X}}(X_j, X_i) = 0$ for all pairs $j, i$. Thus $\mathrm{Ext}(M \oplus \bigoplus_{i=1}^{s-1} X_i, M \oplus \bigoplus_{i=1}^{s-1} X_i) = 0$. We show that $M \oplus \bigoplus_{i=1}^{s-1} X_i$ is a tilting sheaf. For this it remains to prove that if $Z \in \mathrm{coh}(\mathbb{X})$ satisfies $\mathrm{Hom}_{\mathbb{X}}(M \oplus \bigoplus_{i=1}^{s-1} X_i, Z) = 0 = \mathrm{Ext}^1_{\mathbb{X}}(M \oplus \bigoplus_{i=1}^{s-1} X_i, Z)$ then $Z = 0$. We have Auslander-Reiten sequences

$$0 \to N_i \to M^{u_i} \to X_i \to 0.$$

Indeed, since $(N_i, M)$ is an exceptional pair in $\mathcal{C}_i$, $N_i$ is a direct predecessor of $M$ in the Auslander-Reiten quiver. We already have shown that $u_i = n_i$. Finally, because $M$ is injective non-simple, the third term of the Auslander-Reiten sequence is the simple injective $X_i$.

Applying the functor $\mathrm{Hom}_{\mathbb{X}}(-, Z)$ to these sequences our assumptions for $Z$ imply that $\mathrm{Hom}(N_i, Z) = 0 = \mathrm{Ext}^1_{\mathbb{X}}(N_i, Z)$ for all i. Now our claim follows from the fact that $M \oplus \bigoplus_{i=1}^{s-1} N_i$ is a tilting bundle in $\mathrm{coh}(\mathbb{X})$.

We have obtained a tilting sheaf on $\mathbb{X}$ containing only one vector bundle as a direct summand. Applying the concept of perpendicular categories [3] we get a tilting bundle on the usual weighted projective line containing only one vector bundle as direct summand which is impossible. □

**3.6.** We do not know whether $X_i$ can be chosen as a vector bundle. However we have the following result.

**Proposition 3.6.** *If the indecomposables $N_i$ of the Bongartz complement of $M$ are ordered in such a way that $(N_1, N_2, \ldots, N_{s-1}, M)$ form an exceptional sequence then the simple injective module $X$ in $\mathcal{C}(N_{s-1}, M)$ is either a vector bundle or a simple object in* $\mathrm{coh}_0(\mathbb{X})$.

**Proof.** We denote $E_i = \tau_{\mathbb{X}}^{-} N_i$ for $i = 1, \ldots, s-2$. Assume that $X$ is not in $\mathrm{vect}(\mathbb{X})$ and is not simple in $\mathrm{coh}_0(\mathbb{X})$. Let $Z$ be the quasi-top of $X$. Then we have an exact sequence $(\star)$ $\quad 0 \to W \to X \to Z \to 0$ in $\mathrm{coh}_0(\mathbb{X})$. Applying Serre duality the fact that $(N_1, N_2, \ldots, N_{s-1}, M)$ is an exceptional sequence implies that $N_{s-1}, M, E_1, \ldots E_{s-2}$ is an exceptional sequence in $\mathrm{coh}(\mathbb{X})$. Moreover it follows from [6, Proposition 2.8] that the full subcategory generated by $N_{s-1}, M$ coincides with the right perpendicular category $(E_1, \ldots, E_{s-2})^{\perp}$.

Applying the functor $\mathrm{Hom}(E_i, -)$ to the sequence $(\star)$ we get an exact sequence $0 = \mathrm{Hom}_{\mathbb{X}}(E_i, X) \to \mathrm{Hom}_{\mathbb{X}}(E_i, Z) \to \mathrm{Ext}^1_{\mathbb{X}}(E_i, W) \to \mathrm{Ext}^1_{\mathbb{X}}(E_i, X) = 0$. Note that $Z$ does not belong to $(E_1, \ldots, E_{s-2})^{\perp}$ because otherwise $X$ is not simple injective in this category. But all $\mathrm{Ext}^1_{\mathbb{X}}(E_i, Z) = 0$ for all $i$ hence we conclude that $\mathrm{Hom}_{\mathbb{X}}(E_i, Z) \neq 0$ for some $i$. Thus $0 \neq \mathrm{Ext}^1_{\mathbb{X}}(E_i, W) \simeq \mathrm{D}\,\mathrm{Hom}(W, \tau E_i)$ which is impossible because $W$ is a finite length sheaf and $E_i$ a vector bundle. □

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