HOMOLOGICAL METHODS IN THE GENERALIZATION OF DRINFELD MODULES

DAWID E. KĘDZIERSKI, PIOTR KRASOŃ*

Abstract. We introduce and study a natural class of Anderson **t**-modules, called triangular **t**-modules, characterized by having Drinfeld modules as their τ-composition factors. They form a homologically meaningful generalization of Drinfeld modules and exhibit rich arithmetic structure.

We establish criteria for purity, strict and almost strict, and develop a reduction procedure that lowers the degrees of the defining biderivations. As a consequence, every almost strictly pure triangular **t**-module becomes strictly pure after a finite base extension.

We then investigate morphisms and isogenies between triangular **t**-modules, provide a characterization of triangular isogenies, and describe the algebra of endomorphisms, including a criterion for commutativity. On the analytic side, we show that all triangular **t**-modules are uniformizable and establish finiteness and purity criteria with consequences for Taelman's conjecture.

Finally, we develop a duality theory for triangular **t**-modules and their biderivations, proving compatibility with τ-composition series and establishing analogues of the Cartier–Nishi theorem and the Weil–Barsotti formula.

Contents

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*Corresponding author.

1. Introduction

Drinfeld modules and their higher-dimensional analogues, the Anderson **t**-modules, play a central role in the arithmetic of global function fields. This is due to their numerous applications, for instance in class field theory, the theory of automorphic forms, the Langlands program, and Diophantine geometry (see [G96], [Th04], [BP20]). After adjoining the zero object, the category of **t**-modules becomes an $\mathbb{F}_q[t]$-linear additive category, although it is not abelian. Despite many analogies with the category of abelian varieties, one of the key differences is that the category of **t**-modules is far from semisimple, making it natural to study the groups $\mathrm{Ext}^1(\Phi, \Psi)$. The investigation of these extension groups was initiated in [PR03] and subsequently developed in [KK24], [GKK24], and [KK26]. A central technical tool in these works is to interpret extensions as appropriate biderivations. An interesting and subtle question is when $\mathrm{Ext}^1(\Phi, \Psi)$ itself carries a natural **t**-module structure.

In [GKK24] we introduced an algorithm, called the $\mathbf{t}$-*module reduction* algorithm, which allows one to determine $\mathbf{t}$-module structures on $\mathrm{Ext}^1(\Phi, \Psi)$ for wide classes of pairs of $\mathbf{t}$-modules. In that paper the notion of a τ-composition series was developed, and it was proved in [GKK24, Theorem 5.2, Corollary 5.3] that, under suitable hypotheses, the $\mathbf{t}$-module reduction algorithm applies to $\mathbf{t}$-modules possessing specific τ-composition series. Among these, the $\mathbf{t}$-modules whose τ-composition factors are Drinfeld modules deserve special attention. In a suitable basis, such $\mathbf{t}$-modules have triangular matrix form, and hence we refer to them as *triangular* $\mathbf{t}$-*modules.* They form a new and rich class in the category of $\mathbf{t}$-modules and, from the homological viewpoint, they generalize the class of Drinfeld modules. This leads to the following question:

Question 1.1. *Which properties of Drinfeld modules are inherited by triangular* $\mathbf{t}$*-modules?*

The goal of this paper is to answer this question, as well as related questions for the corresponding $\mathbf{t}$-motives and $\mathbf{t}$-comotives associated with triangular $\mathbf{t}$-modules.

In Section 2, we recall the basic definitions and results needed later in the paper. In particular, we review both the analytic and the algebraic approaches to Anderson $\mathbf{t}$-modules, and discuss extensions of $\mathbf{t}$-modules via biderivations. We also introduce the category of $\mathbf{t}$-motives, which is known to be anti-equivalent to the category of $\mathbf{t}$-modules (cf. Theorem 2.1), as well as the dual notion of a $\mathbf{t}$-comotive.

Remark 1.1. Following [GM25] we use the term $\mathbf{t}$-*comotive* instead of the more traditional *dual* $\mathbf{t}$-*motive*. This terminology avoids potential confusion with the dual of a $\mathbf{t}$-motive in the usual categorical sense.

In Section 3, we prove that the category of $\mathbf{t}$-modules is exact, as defined by Quillen in [Q73]. While it is well known that the category of abelian $\mathbf{t}$-motives is exact, we work directly inside the category of $\mathbf{t}$-modules, describing explicitly the admissible monomorphisms and epimorphisms (see Section 3).

In Section 4, we introduce triangular t-modules and give necessary and sufficient conditions (see Proposition 4.5) for such modules to be strictly pure or almost strictly pure. We then develop a reduction theory culminating in a reduced form theorem (see Proposition 4.1), which shows that any triangular t-module acquires a form with biderivations of minimal degree after a finite base extension of a controlled type.

Section 5 is devoted to morphisms between triangular $\mathbf{t}$-modules. We introduce the notion of a triangular morphism, which behaves analogously to morphisms of Drinfeld modules; see Proposition 5.4. We then study triangular isogenies and give a characterization of when a triangular morphism is an isogeny (cf. Proposition 5.5). We conclude this section by studying the algebra of endomorphisms and provide a sufficient condition for its commutativity. Proposition 5.9 concerns the effective determination of morphisms in a finite characteristic.

In Section 6, we develop the analytic theory of triangular $\mathbf{t}$-modules. In particular, we prove a general result (Theorem 6.2) on the behavior of exponential functions in exact sequences. As a consequence, every triangular $\mathbf{t}$-module is uniformizable (see Corollary 6.6).

Section 7 shows that triangular $\mathbf{t}$-modules are $\mathbf{t}$-finite and therefore abelian. Theorem 7.5 provides a criterion for purity, and we conclude the section with corollaries concerning Taelman's conjecture for triangular $\mathbf{t}$-modules (see Corollaries 7.7 and 7.8).

In Section 8, we recall the definition of the dual $\mathbf{t}$-module introduced in [KK26] and study the duality of triangular $\mathbf{t}$-modules. We prove that this duality is compatible with τ-composition series and also establish the Weil–Barsotti formula (Theorem 8.2). We then adapt duality to biderivations, defining the dual biderivation $\delta^{\vee}$ and its double dual $(\delta^{\vee})^{\vee}$, and show that the assignments $\delta \mapsto \delta^{\vee}$ and $\delta^{\vee} \mapsto (\delta^{\vee})^{\vee}$ are $\mathbb{F}_q$-linear (Corollary 8.8). Finally, we give explicit formulas for dual biderivations in the case of triangular $\mathbf{t}$-modules allowing for LD-biderivations.

Section 9 establishes an analog of the Cartier–Nishi theorem for triangular $\mathbf{t}$-modules allowing for LD-biderivations (Theorem 9.3). We also prove Corollaries 9.4 and 9.5, which describe cases where the sequence of ranks of sub-quotients is strictly increasing or at least non-decreasing. As in [KK26], we use the Ext-definition of the dual $\mathbf{t}$-module. In [KK26, Appendix] by refining the methods of [T95], it was shown, that for strictly pure $\mathbf{t}$-modules without nilpotence the dual module of Taguchi is isomorphic to that given by the Ext functor. Since triangular $\mathbf{t}$-modules are not strictly pure in general, our results provide a new class of $\mathbf{t}$-modules for which the dual module exists and satisfies $(\Phi^{\vee})^{\vee} \cong \Phi$.

2. Background

Let p be a prime number, $\mathbb{F}_q$ a finite field with $q = p^s$ elements, and $A = \mathbb{F}_q[t]$ the ring of polynomials in one variable. By k we denote the field of rational functions $\mathbb{F}_q(\theta)$ in a variable θ. The $\infty-$adic completion $k_\infty := \mathbb{F}_q((1/\theta))$ can be given a normalized valuation $|\cdot|_\infty$ satisfying $|\theta|_\infty = q$. Finally, in order to ensure that the field under consideration is algebraically closed and complete, one has to pass to $\mathbb{C}_\infty$ the completion with respect to $|\cdot|_\infty$ of the algebraic closure $\overline{k_\infty}$. Then there exists a natural embedding $\iota : A \longrightarrow \mathbb{C}_\infty$.

2.1. **Drinfeld modules.** Recall that an $A-$submodule $\Lambda \subset \mathbb{C}_\infty$ is called an $A-$lattice if it is a finitely generated, projective $A-$module, discrete in the topology of $\mathbb{C}_\infty$. For every $A-$lattice Λ there is an entire function: $\exp_\Lambda : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ defined by the formula:

$$\exp_\Lambda(z) = z \prod_{0\neq\lambda\in\Lambda} \Big(1 - \frac{z}{\lambda}\Big).$$

The function $\exp_\Lambda$ is $\mathbb{F}_q$-linear and induces the following short, exact sequence of $A-$modules:

$$0 \longrightarrow \Lambda \longrightarrow \mathbb{C}_\infty \xrightarrow{\exp_\Lambda} \mathbb{C}_\infty \longrightarrow 0 .$$

Let $\mathbb{G}_a$ be an additive algebraic group defined over $\mathbb{C}_\infty$. Then $\mathrm{End}(\mathbb{G}_a) = \mathbb{C}_\infty\{\tau\}$, where $\mathbb{C}_\infty\{\tau\}$ is an $\mathbb{F}_q$-algebra of twisted polynomials, i.e. which fulfills the condition $\tau z = z^q\tau$ for $z \in \mathbb{C}_\infty$. V. Drinfeld showed that for every $a \in A$ there is a twisted polynomial $\phi_a \in \mathbb{C}_\infty\{\tau\}$ such that the following functional equation:

$$\exp_\Lambda(az) = \phi_a(\exp_\Lambda(z))$$

holds true. The map $a \mapsto \phi_a$ is an $\mathbb{F}_q-$algebra homomorphism called the Drinfeld module associated with Λ. We will denote it as $\phi_\Lambda : A \longrightarrow \mathbb{C}_\infty\{\tau\}$. The rank of a Drinfeld module is the rank of the lattice Λ and will be denoted as $\mathrm{rk}\,\phi_\Lambda$. Notice that the Drinfeld module $\phi_\Lambda : A \longrightarrow \mathbb{C}_\infty\{\tau\}$ is uniquely determined by its value at t:

$$\phi_t = \theta + a_1\tau + a_2\tau^2 + \cdots a_n\tau^n.$$

Also $n = \deg_\tau \phi_t$ is equal to the rank of ϕ_Λ. More generally, one has $\deg_\tau \phi_a = \deg_t a \cdot \mathrm{rk}\,\phi_\Lambda$ for every $a \in A$.

2.2. **Anderson t-modules.** Higher dimensional analogs of Drinfeld modules are Anderson **t**-modules [A86]. They are defined as $\mathbb{F}_q-$algebra homomorphisms:

$$\Phi : A \longrightarrow \mathrm{Mat}_d(\mathbb{C}_\infty\{\tau\}) \tag{2.1}$$

where

$$\Phi_t = \partial\Phi_t + A_1\tau + A_2\tau^2 + \cdots A_n\tau^n,$$

and ∂ denotes a derivative on $\mathbb{G}_a^d$. Here $\partial\Phi_t = \theta I_d + N_\Phi$ where I_d is the identity matrix and N_Φ is a nilpotent matrix. We say that Φ has dimension d and is of degree $\deg\Phi := \deg_\tau \Phi_t$. In this case there also exists a uniquely determined exponential function $\mathrm{Exp}_\Phi : \mathbb{C}_\infty^d \longrightarrow \mathbb{C}_\infty^d$, defined by the series:

$$\mathrm{Exp}_\Phi(\tau) = I_d\tau^0 + \sum_{i=1}^{\infty} B_i\tau^i$$

satisfying the following functional equation:

$$\mathrm{Exp}_\Phi\big(\partial\Phi_a\tau\big) = \Phi_a\big(\mathrm{Exp}_\Phi(\tau)\big).$$

Exp_Φ is defined on the whole of $\mathbb{C}_\infty^d$, but does not need to be surjective. If Exp_Φ is surjective then we say that the $\mathbf{t}$-module Φ is uniformizable. For example, Drinfeld modules are one-dimensional, uniformizable $\mathbf{t}$-modules. Recall that Exp_Φ is a homomorphism of $\mathbb{F}_q[t]-$modules and its kernel Λ is a discrete finitely generated $\partial\Phi(A)-$ submodule of $\mathbb{C}_\infty^d$. We call Λ a $\partial\Phi(A)-$lattice for Φ. Its rank is defined to be the rank of Φ. Notice that for a $\mathbf{t}$-module Φ of dimension $d \geq 2$ its degree $\deg\Phi$ is not necessarily equal to $\mathrm{rk}\,\Phi$ (cf. [G96, Example 5.9.9]).

Considering $\mathbf{t}$-modules in terms of the exponential functions and lattices yields an analytic approach to the theory. Here, one has to work with the field $\mathbb{C}_\infty$ or at least a sufficiently large subfield (containing the lattice of Φ). If we consider a $\mathbf{t}$-module as a homomorphism (2.1) then we have an algebraic approach to the theory. In this case we can consider $\mathbf{t}$-modules for an arbitrary $A-$field K, i.e. a field of characteristic p equipped with a homomorphism $\iota : A \longrightarrow K$ such that $\iota(t) := \theta$. We say that an $A-$field K is of $A-$ characterisic zero if ι is an inclusion. If ι factors as $A \longrightarrow A/\mathfrak{p} \longrightarrow K$ where $\mathfrak{p}$ is a prime ideal of A then we say that the $A-$characteristic of K is $\mathfrak{p}$. We will denote the $A-$ characterisic of K as $\mathrm{char}_A K$. In this paper we are usually considering $\mathbf{t}$-modules defined over an arbitrary $A-$field K. The only exception is section 6, in which we investigate analytic properties of triangular $\mathbf{t}$-modules. So, we will work with the field $\mathbb{C}_\infty$ therein.

Recall that $\mathbf{t}$-modules form a category. A morphism between $\mathbf{t}$-modules Φ and Ψ of dimensions d and e, respectively, is given by the matrix $F \in \mathrm{Mat}_{e\times d}\big(K\{\tau\}\big)$ satisfying the following condition:

$$F\Phi = \Psi F.$$

After one adjoins the zero $\mathbf{t}$-module $0 : A \longrightarrow 0$ this category becomes additive but nonabelian. The category of $\mathbf{t}$-modules will be denoted as $\mathbf{t}$-mod, and the space of morphisms as $\mathrm{Hom}_\tau(\Phi, \Psi)$.

2.3. **Extension of t-modules.** Every $\mathbf{t}$-module Φ of dimension d induces an $A-$module structure on the space K^d. This is given by the following formula:

$$a * x := \Phi_a(x), \quad \text{for} \quad a \in A, x \in K^d.$$

This module is called the Mordell-Weil group of Φ and is denoted as $\Phi(K^d)$. Analogously, every morphism of $\mathbf{t}$-modules induces a morphism of $A-$modules. In this way one obtains the inclusion functor from the category $\mathbf{t}$-mod into the category $A-\mathrm{mod}$. This functor enables us to transfer the notion of exact sequences to the category of $\mathbf{t}$-modules in the following way. We say that the sequence of $\mathbf{t}$-modules

$$0 \longrightarrow \Phi \longrightarrow \Upsilon \longrightarrow \Psi \longrightarrow 0 \tag{2.2}$$

is exact if the corresponding sequence of $A-$modules

$$0 \longrightarrow \Phi(K^d) \longrightarrow \Upsilon(K^{e+d}) \longrightarrow \Psi(K^e) \longrightarrow 0$$

is exact. By $\mathrm{Ext}^1_\tau(\Psi, \Phi)$ we will denote the set of exact sequences of the form (2.2) modulo the isomorphic sequences. Similarly, to the classical situation of modules over algebras the set $\mathrm{Ext}^1_\tau(\Psi, \Phi)$ is an $\mathbb{F}_q-$linear vector space. It is also an $A-$module where multiplication by a is given by the pullback by Ψ_a, or equivalently the pushout by Φ_a.

For a short, exact sequence of $\mathbf{t}$-modules there exists a six-term $\mathrm{Hom}-\mathrm{Ext}$ exact sequence , (see[KK24, Theorem 10.2]) for both the domain

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_\tau(\zeta, \Phi) \longrightarrow \mathrm{Hom}_\tau(\zeta, \Upsilon) \longrightarrow \mathrm{Hom}_\tau(\zeta, \Psi) \longrightarrow \\ \longrightarrow \mathrm{Ext}^1_\tau(\zeta, \Phi) \longrightarrow \mathrm{Ext}^1_\tau(\zeta, \Upsilon) \longrightarrow \mathrm{Ext}^1_\tau(\zeta, \Psi) \longrightarrow 0, \end{aligned}$$

and also for the range of the Hom_τ.

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_\tau(\Psi, \zeta) \longrightarrow \mathrm{Hom}_\tau(\Upsilon, \zeta) \longrightarrow \mathrm{Hom}_\tau(\Phi, \zeta) \longrightarrow \\ \longrightarrow \mathrm{Ext}^1_\tau(\Psi, \zeta) \longrightarrow \mathrm{Ext}^1_\tau(\Upsilon, \zeta) \longrightarrow \mathrm{Ext}^1_\tau(\Phi, \zeta) \longrightarrow 0. \end{aligned}$$

In this paper we will use a description of exact sequences by biderivations. This approach was suggested by M.A. Papanikolas and N. Ramachandran in [PR03]. According to this, by an appropriate choice of coordinates in the $\mathbf{t}$-module Υ in (2.2), we can assume that the maps in this sequence are an inclusion on the second coordinate followed by projection onto the first coordinate, thereby we obtain the following sequence:

$$0 \longrightarrow \Phi \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \Upsilon \xrightarrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} \Psi \longrightarrow 0.$$

Then Υ is given by the following block matrix:

$$\left[\begin{array}{c|c} \Psi & 0 \\ \hline \delta & \Phi \end{array}\right] : \mathbb{F}_q[t] \longrightarrow \mathrm{Mat}_{e+d}(K\{\tau\}).$$

Therefore each extension is given by the $\mathbb{F}_q-$linear map

$$\delta : \mathbb{F}_q[t] \longrightarrow \mathrm{Mat}_{d\times e}(K\{\tau\})$$

such that

$$\delta_{ab} = \delta_a \Psi_b + \Phi_a \delta_b \quad \text{for each} \quad a, b \in \mathbb{F}_q[t]$$

where δ_a is an evaluation of δ at a. The space of biderivations will be denoted as $\mathrm{Der}(\Psi, \Phi)$. Two exact sequences given by the biderivations δ and $\widehat{\delta}$ are isomorphic if there is a matrix $U \in \mathrm{Mat}_{d\times e}(K\{\tau\})$ such that

$$\delta - \widehat{\delta} = U\Psi - \Phi U.$$

The space of biderivations of the form $\delta^{(U)} = U\Psi - \Phi U$ will be denoted as $\mathrm{Der}_{in}(\Psi, \Phi)$ and its elements will be called inner biderivations. Then we have the following isomorphism of $A-$modules

$$\mathrm{Ext}^1_\tau(\Psi, \Phi) \cong \mathrm{Der}(\Psi, \Phi)/\mathrm{Der}_{in}(\Psi, \Phi). \tag{2.3}$$

In the sequel, we will denote an exact sequence by

$$\delta : \quad 0 \longrightarrow \Phi \longrightarrow \Upsilon \longrightarrow \Psi \longrightarrow 0,$$

if it is given by the biderivation δ. Notice that every biderivation δ is uniquely determined by its value at t. This yields an isomorphism of the $\mathbb{F}_q-$linear spaces: $\mathrm{Der}(\Psi, \Phi) \cong \mathrm{Mat}_{d\times e}(K\{\tau\})$ given by the assignment $\delta \mapsto \delta_t$. Recall that the $\mathbb{F}_q[t]$-module action on $\mathrm{Ext}^1_\tau(\Psi, \Phi)$ on the level of biderivations is realized by means of the following formula:

$$t * \Big(\delta + \mathrm{Der}_{in}(\Psi, \Phi)\Big) := \Phi_t \delta. + \mathrm{Der}_{in}(\Psi, \Phi).$$

In the sequel, to simplify the notation we omit $+\mathrm{Der}_{in}(\Psi, \Phi)$ when we consider the class $\delta + \mathrm{Der}_{in}(\Psi, \Phi)$.

2.4. **Anderson t-motives.** For an A-field K let $K[t, \tau] := K\{\tau\}[t]$ be the polynomial ring over $K\{\tau\}$ with the following commutation relations:

$$tc = ct, \ t\tau = \tau t, \ \tau c = c^q \tau, \quad c \in K.$$

Definition 2.1 ([A86])**.** A $\mathbf{t}$-motive M is a left $K[t, \tau]$-module which is free and finitely generated as a $K\{\tau\}$-module for which there is an $l \in \mathbb{N}$ with

$$(t - \theta)^l (M/\tau M) = 0.$$

A morphism of $\mathbf{t}$-motives is a morphism of $K[t, \tau]$-modules.

With every $\mathbf{t}$-module (Φ, G_a^d) over K one uniquely associates a $\mathbf{t}$-motive in the following way: $M(\Phi) = \mathrm{Hom}_K^q(G_a^d, G_a) \cong K\{\tau\}^d$ is the set of $\mathbb{F}_q$-linear morphisms of algebraic groups endowed with the following $K[t,\tau]$-action $(ct^i, m) \to c \circ m \circ \Phi(t^i)$.

Definition 2.2. A $\mathbf{t}$-motive M is called abelian if M is free and finitely generated over $K[t]$. Its rank $\mathrm{rk}\, M$ as a $K[t]$-module is called the rank of M. An abelian t-module is a $\mathbf{t}$-module for which the associated $\mathbf{t}$-motive is abelian.

The following theorem was proved by G. Anderson [A86] (see also [vH04, Theorem 2.9], [BP20]).

Theorem 2.1. *The above correspondence between abelian* $\mathbf{t}$*-motives and abelian* $\mathbf{t}$*-modules over* K *gives an anti-equivalence of categories.*

In order to have the category $\mathcal{C}$ of motives which is closed with respect to tensor products and sub-quotients, one introduces the notion of pure $\mathbf{t}$-motives. Set $M((\frac{1}{t})) := M \otimes_{K[t]} K((\frac{1}{t}))$ where $K[[\frac{1}{t}]]$ is the ring of power series in the variable $1/t$ and $K((\frac{1}{t}))$ is its field of fractions. $M((\frac{1}{t}))$ is naturally a left $K[t,\tau]$-module. If $M((\frac{1}{t}))$ contains a finitely generated $K[[\frac{1}{t}]]$-submodule H, which generates $M((\frac{1}{t}))$ over $K((\frac{1}{t}))$ and additionally satisfies

$$t^u H = t^v H$$

for some $u, v \in \mathbb{N}$, then M is called pure of weight $w(M) = \frac{d(M)}{\mathrm{rk}(M)} = \frac{u}{v}$. The category of pure $\mathbf{t}$-motives is closed with respect to tensor products and sub-quotients [A86],[vH04, section 3]. In particular $\Lambda^k M$ is pure for any pure $\mathbf{t}$-motive M and $k \in \mathbb{N}$. Moreover, [vH04, Proposition 3.22 (i)] asserts that if a pure $\mathbf{t}$-motive sits in an exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

of $\mathbf{t}$-motives, then M_1 and M_2 are also pure and one has

$$w(M_1) = w(M_2) = w(M) \tag{2.4}$$

We will use this fact in section 8.

2.5. **t-comotives.** Let L be an extension of $\mathbb{F}_q$ which is a perfect field. Let $L[t,\sigma]$ be a polynomial ring with coefficients in L subject to the following relations:

$$ct = tc, \quad \sigma t = t\sigma, \quad \sigma c = c^{1/q}\sigma, \qquad c \in L^*.$$

Definition 2.3. A $\mathbf{t}$-comotive N is a left $L[t,\sigma]$-module that is finitely generated and free over $L\{\sigma\}$, such that there exists $l \in \mathbb{N}$ with

$$(t-\theta)^l(N/\sigma N) = 0.$$

A morphism of $\mathbf{t}$-comotives is a morphism of $L[t,\sigma]$-modules.

With every $\mathbf{t}$-module (Φ, G_a^d) over L one uniquely associates a $\mathbf{t}$-comotive in the following way:

Let $* : L\{\tau\} \to L\{\sigma\}$ be an anti-isomorphism defined as follows: $(\sum a_i\tau^i)^* = \sum \sigma^i a_i = \sum a_i^{(-i)}\sigma^i$. Extend this map to $* : \mathrm{Mat}_{n\times m}L\{\tau\} \to \mathrm{Mat}_{m\times n}(L\{\sigma\})$ by defining $(B^*)_{i,j} = B^*_{j,i}$.

$N(\Phi) = \mathrm{Hom}_K^q(G_a, G_a^d) \cong L\{\sigma\}^d$ is the set of $\mathbb{F}_q$-linear morphisms of algebraic groups endowed with the following $L[t,\sigma]$-action

$$(ct^i, h) \to c \circ h \circ \Phi(t^i)^*, \quad \text{where} \quad h \in N(\Phi).$$

The following theorem was proved G. Anderson (see [BP20]).

Theorem 2.2. *Over a perfect field L, the above correspondence between* $\mathbf{t}$*-comotives and* $\mathbf{t}$*-modules gives an equivalence of categories.*

Definition 2.4. A $\mathbf{t}$-comotive M is called finite if M is free and finitely generated over $K[t]$. A finite $\mathbf{t}$-module is a $\mathbf{t}$-module for which the associated dual $\mathbf{t}$-motive is finite.

Recently A.Maurischat [M21] has clarified the situation and proved that a $\mathbf{t}$-module is abelian iff it is finite. We will use his result in section 7.

3. Exact category structure

Recall that an exact category is an additive category $\mathcal{A}$ with the fixed class E of kernel-cokernel pairs (i, d), i.e. they are pairs of composable morphisms such that i is a kernel of d and d is a cokernel of i, (c.f. [Q73]). We will call a morphism i, for which there exists a morphism d, such that $(i, d) \in E$, an admissible monomorphism. Similarly, a morphism d for which there exists i such that $(i, d) \in E$, will be called an admissible epimorphism. We require that E is closed under isomorphisms and satisfies the following axioms:

(i) For all objects $A \in \mathcal{OA}$ the identity 1_A is an admissible monic,
(ii) For all objects $A \in \mathcal{OA}$ the identity 1_A is an admissible epic,
(iii) The class of admissible monics is closed under composition,
(iv) The class of admissible epics is closed under composition,

(v) The pushout of an admissible monic $i : A \rightarrowtail B$ along an arbitrary morphism $f : A \to C$ exists and yields an admissible monic j:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r, tail, "j"] & D \end{tikzcd} . \tag{3.1}$$

(v) The pullback of an admissible epic $h : C \twoheadrightarrow D$ along an arbitrary morphism $g : B \to D$ exists and yields an admissible epic k:

$$\begin{tikzcd} A \arrow[r, two heads, "k"] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r, two heads, "h"] & D \end{tikzcd} \tag{3.2}$$

Proposition 3.1. *The category of* $\mathbf{t}$*-modules with the set* E *of admissible pairs* (i, d) *coming from the* $\tau-$*exact sequences:*

$$0 \longrightarrow \Psi \overset{i}{\rightarrowtail} \Upsilon \overset{d}{\twoheadrightarrow} \Psi \longrightarrow 0 \tag{3.3}$$

is an exact category.

Proof. For any object $A \in \mathcal{A}$, the exact sequence $0 \longrightarrow A \overset{1_A}{\rightarrowtail} A \twoheadrightarrow 0 \longrightarrow 0$ (resp. $0 \longrightarrow 0 \rightarrowtail A \overset{1_A}{\twoheadrightarrow} A \longrightarrow 0$) shows that 1_A is an admissible monic (resp. epic). This proves (i) and (ii).

In order to prove that a composition of two admissible monomorphisms $i_2 \circ i_1$ where $i_1 : A \rightarrowtail B$ and $i_2 : B \rightarrowtail C$ is admissible, consider the following block triangular matrices corresponding to i_1 (resp. i_2): $B = \left[\begin{array}{c|c} B/A & 0 \\ \hline \delta_1 & A \end{array}\right]$ (resp. $C = \left[\begin{array}{c|c} C/B & 0 \\ \hline \delta_2 & B \end{array}\right]$ Notice that any extension can be described by a block lower triangular matrix (in a suitable basis) [GKK24, proof of Theorem 5.1]

Then consider the extension corresponding to the following matrix:

$$\widehat{C} = \left[\begin{array}{c|c||c} C/B & 0 & 0 \\ \hline \delta_{2a} & B/A & 0 \\ \hline\hline \delta_{2b} & \delta_1 & A \end{array}\right] \quad \text{where} \quad \delta_2 = \left[\begin{array}{c} \delta_{2a} \\ \hline \delta_{2b} \end{array}\right].$$

From the form of the above matrix we obtain the following extension of $\mathbf{t}$-modules:

$$0 \longrightarrow A \overset{i_2 \circ i_1}{\rightarrowtail} C \overset{d}{\twoheadrightarrow} D \longrightarrow 0 \tag{3.4}$$

where D is given by $\left[\begin{array}{c|c} C/B & 0 \\ \hline \delta_{2a} & B/A \end{array}\right]$. For the composition of two admissible epics we proceed analogously. Let $d_1 : A \twoheadrightarrow B$ and $d_2 : B \twoheadrightarrow C$ be admissible epics. Let $A = \left[\begin{array}{c|c} B & 0 \\ \hline \delta_1 & D \end{array}\right]$ (resp. $B = \left[\begin{array}{c|c} C & 0 \\ \hline \delta_2 & E \end{array}\right]$) correspond to d_1 (resp. d_2), then the following matrix:

$$\widehat{A} = \left[\begin{array}{c||c|c} C & 0 & 0 \\ \hline\hline \delta_2 & E & 0 \\ \hline \delta_{1b} & \delta_{1a} & A \end{array}\right] \quad \text{where} \quad \delta_1 = \left[\begin{array}{c|c} \delta_{1b} & \delta_{1a} \end{array}\right]$$

shows that $d_2 \circ d_1$ is an admissible epimorphism.

Parts (iv) and (v) are the content of [KK24, Theorem 10.1]. $\square$

4. Triangular modules

In this section we introduce triangular modules and prove their basic properties.

4.1. **τ-composition series and τ-co-composition series.** A concept of composition series for $\mathbf{t}$-modules was introduced in [GKK24]. According to this, a $\mathbf{t}$-module is called $\tau-$simple if it is not a middle term in any nontrivial, short, exact sequence. Then every $\mathbf{t}$-module Υ is either $\tau-$simple or there exists a $\tau-$composition series of the form:

$$0 = \Upsilon_0 \hookrightarrow \Upsilon_1 \hookrightarrow \Upsilon_2 \hookrightarrow \cdots \hookrightarrow \Upsilon_n = \Upsilon, \tag{4.1}$$

where the quotients $\Upsilon_{l+1}/\Upsilon_l$ are $\tau-$simple for $l = 0, 1, \ldots, n-1$, see [GKK24, Theorem 5.1].

Similarly, one can consider $\tau-$co-composition series of the form:

$$0 = \widehat{\Upsilon}_0 \twoheadleftarrow \widehat{\Upsilon}_1 \twoheadleftarrow \widehat{\Upsilon}_2 \twoheadleftarrow \cdots \twoheadleftarrow \widehat{\Upsilon}_n = \Upsilon, \tag{4.2}$$

where $\ker\left(\widehat{\Upsilon}_l \twoheadrightarrow \widehat{\Upsilon}_{l-1}\right)$, $\quad l = n, n-1, \ldots, 1$ are $\tau-$simple $\mathbf{t}$-modules.

Definition 4.1. We say that a $\mathbf{t}$-module Υ is triangular if it is given by the τ-composition series (4.1) where the sub-quotients $\Upsilon_{l+1}/\Upsilon_l$ are Drinfeld modules for $l = 0, 1, \ldots, n-1$.

Remark 4.2. Let Υ be a triangular $\mathbf{t}$-module given by the composition series (4.1). Denote $\psi_{l+1} := \Upsilon_{l+1}/\Upsilon_l$ for $l = 0, 1, 2, \ldots, n-1$. The $\tau-$composition series (4.1) implies existence of the following exact sequences:

$$\delta_l: \quad 0 \longrightarrow \Upsilon_l \longrightarrow \Upsilon_{l+1} \longrightarrow \psi_{l+1} \longrightarrow 0 \quad \text{for} \quad l = 1, 2, \ldots, n-1,$$

where $\Upsilon_1 = \psi_1$ is a Drinfeld module. Then

(1) the matrix Υ_t is of the following form:

$$
\Upsilon_t = \left(\begin{array}{cccccc}
\boxed{\psi_n} & 0 & \cdots & 0 & 0 & 0\\
 & \boxed{\psi_{n-1}} & \cdots & 0 & 0 & 0\\
 & & \ddots & \vdots & \vdots & \vdots\\
\delta_{n-1} & \delta_{n-2} & & \boxed{\psi_3} & 0 & 0\\
 & & \cdots & \delta_2 & \boxed{\psi_2} & 0\\
 & & & & \delta_1 & \boxed{\psi_1}
\end{array}\right), \tag{4.3}
$$

(2) Υ is determined by the pair $\left(\underline{\psi}, \underline{\delta}\right)$, where

$$
\underline{\psi} = \left(\psi_1, \psi_2, \ldots, \psi_n\right) \quad \text{and} \quad \underline{\delta} = \left(\delta_1, \delta_2, \cdots, \delta_{n-1}\right).
$$

Conversely, every pair $\left(\underline{\psi}, \underline{\delta}\right)$ determines the $\mathbf{t}$-module Υ. The fact that the triangular $\mathbf{t}$-module Υ is determined by the pair $\left(\underline{\psi}, \underline{\delta}\right)$ will be denoted in the following way: $\Upsilon = \Upsilon\left(\underline{\psi}, \underline{\delta}\right)$.

(3) Notice that δ_1 is a skew polynomial and appears in the matrix Υ_t in the same row as ψ_1 and in the same column as ψ_2. So, we can write δ_1 in the following way:

$$
\delta_1(\tau) = \left[\delta_{1\times 2}(\tau)\right] = \left[\sum_{k=0}^{\deg \delta_1} d_{1\times 2,k}\tau^k\right]
$$

Using analogous convention we can write

$$
\delta_2(\tau) = \begin{bmatrix} \delta_{2\times 3}(\tau) \\ \delta_{1\times 3}(\tau) \end{bmatrix} = \begin{bmatrix} \sum\limits_{k=0}^{\deg \delta_2} d_{2\times 3,k}\tau^k \\ \sum\limits_{k=0}^{\deg \delta_2} d_{1\times 3,k}\tau^k \end{bmatrix}.
$$

and in general

$$
\delta_l(\tau) = \begin{bmatrix} \delta_{l\times l+1}(\tau) \\ \delta_{l-1\times l+1}(\tau) \\ \vdots \\ \delta_{1\times l+1}(\tau) \end{bmatrix} = \begin{bmatrix} \sum\limits_{k=0}^{\deg \delta_l} d_{l\times l+1,k}\tau^k \\ \sum\limits_{k=0}^{\deg \delta_l} d_{l-1\times l+1,k}\tau^k \\ \vdots \\ \sum\limits_{k=0}^{\deg \delta_l} d_{1\times l+1,k}\tau^k \end{bmatrix} \tag{4.4}
$$

for $l = 1, 2, \ldots, n-1$. Notice that since degrees $\delta_{j\times l+1}$ are usually not equal, some of the coefficients $d_{j\times l+1,k}$ might be zeros.

4.2. **t-reduded form.** We now turn to the problem of determining a reduced form for triangular **t**-modules. In general, such a form will not exist over the original base field K. It might only exist after a suitable finite extension of K.

Theorem 4.1. *Let $\Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big)$ be a triangular* **t***-module of dimension n defined over a field K. There exists a finite extension $K(\Upsilon)/K$, such that Υ admits a reduced form $\Upsilon^{\mathrm{red}} = \Upsilon\big(\underline{\psi}^{\mathrm{red}}, \underline{\delta}^{\mathrm{red}}\big)$ over $K(\Upsilon)$ satisfying*

$$\deg_\tau \delta^{\mathrm{red}}_{i\times j} \;<\; \max\{\mathrm{rk}\,\psi_i,\, \mathrm{rk}\,\psi_j\} \qquad \textit{for all} \quad i > j.$$

Proof. Let Υ be of the form (4.4). Denote by r_k the rank of the module ψ_k and by a_{k,r_k} its leading term.

Notice that every lower triangular matrix $U \in \mathrm{Mat}_n\big(K\{\tau\}\big)$ with diagonal terms equal to one induces an isomorphism of the **t**-module Υ with a triangular **t**-module $U\cdot\Upsilon\cdot U^{-1}$. Denote the entries in the matrix U as $u_{i\times j}$ in the same way as the biderivations in the **t**-module Υ.

Consider an isomorphism $U_{i\times j}$ given by the matrix, of which below diagonal has only one non-zero entry $u_{i\times j}$. Then

$$U_{i\times j}\cdot\Upsilon\cdot U_{i\times j}^{-1} = \Upsilon + \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ u_{i\times j}\delta_{j\times n} & \cdots & u_{i\times j}\delta_{j\times n} & \delta^{(u_{i\times j})}_{\psi_j,\psi_i} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -u_{i\times j}\delta_{i-1\times i} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -u_{i\times j}\delta_{1\times i} & 0 & \cdots & 0 \end{bmatrix},$$

where $\delta^{(u_{i\times j})}_{\psi_j,\psi_i}$ is an inner biderivation in $\mathrm{Der}_{in}(\psi_j,\psi_i)$. This isomorphism allows one to reduce the biderivation $\delta_{i\times j}$ by means of the inner biderivation $\delta^{(u_{i\times j})}_{\psi_j,\psi_i}$. As a result of this operation, the entries to the left and also below the biderivation $\delta_{i\times j}$ have been changed. Notice that putting $u_{i\times j} = \mu\cdot\tau^k$ for $\mu \in K$ and $k = 0, 1, 2, \ldots$ we obtain the following form for $\delta^{(u_{i\times j})}_{\psi_j,\psi_i}$:

$$\delta^{(\mu\tau^k)}_{\psi_j,\psi_i} = f_k(\mu)\cdot\tau^{k+\max\{r_i,r_j\}} + \text{terms of smaller degree},$$

where $f_k(x) \in K[x]$.

Assume that $\deg_\tau \delta_{i\times j} = h > \max\{r_i, r_j\}$ and that its leading term is equal to b. Setting $k = h - r_j$ and μ as the root of the following equation:

$$f_k(x) + b = 0$$

we determine that the degree of the biderivation at the place $i\times j$ in the **t**-module $U_{i\times j}\cdot\Upsilon\cdot U_{i\times j}^{-1}$ is less than h. If, in the field K, the equation

$f_k(x)+b=0$ has no solution then we go to the extension $K(\mu)$ and proceed further over this new field.

The reduction algorithm proceeds by increasing distance from the diagonal. At the first stage we consider all entries in the first sub-diagonal. Once these are reduced, we move to the second sub-diagonal, and so forth. At the m-th step we simultaneously reduce all entries at distance m from the diagonal, proceeding inductively until all off-diagonal entries have been reduced.

$\square$

Remark 4.3. The field $K(\Upsilon)$ arising in the reduced form of a triangular $\mathbf{t}$-module Φ is obtained by adjoining solutions of certain additive equations of the form

$$a\,x^{q^k} - b\,x + c = 0, \qquad a, b, c \in K,$$

which naturally occur during the reduction process. Depending on whether $a = 0$, or $b = 0$, or both are nonzero, these equations give rise respectively to either linear equations, or purely inseparable Frobenius equations $x^{q^k} = d$, or to additive q^k-polynomial equations which, after a suitable change of variables[1], reduce to the classical Artin–Schreier form

$$y^{q^k} - y = d'.$$

Thus each extension appearing in $K(\Upsilon)$ is a compositum of two fundamental families of extensions in characteristic p: purely inseparable Frobenius extensions, and separable Artin–Schreier (or more generally Artin–Schreier–Witt) extensions. The latter are well understood and have a complete cohomological classification via Witt vectors (see Neukirch [N99, Chap. IV], Goss [G96, §12.1], and Hazewinkel [Haz09]; see also Serre [S79, Chap. III, Ex. 6] for the basic case $k = 1$). Consequently, although the reduction procedure may require adjoining to K several solutions of such equations, the resulting field $K(\Upsilon)$ always lies within the classically studied framework of Frobenius and Artin–Schreier–Witt extensions.

For a triangular $\mathbf{t}$–module Υ we call its reduced form Υ^{red} the *reduced form* of Υ. Let us note that, from the proof of the preceding theorem, one can easily extract a condition on the degrees of the biderivations which guarantees that $K(\Upsilon) = K$, independently of the nature of the base field K.

[1] Such a change of variables may require passing to a finite extension of K.

Corollary 4.2. *Let $\Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big)$ be a triangular* $\mathbf{t}$*-module of dimension n defined over a field K. If the condition*

$$\forall_{i<j} \qquad \deg_\tau \delta_{i\times j} \ \geq \ \max\{\operatorname{rk}\psi_i, \operatorname{rk}\psi_j\} \quad \Longrightarrow \quad \operatorname{rk}\psi_j \ > \ \operatorname{rk}\psi_i \tag{4.5}$$

is satisfied, then $K(\Upsilon) = K$.

Proof. Condition (4.5) implies that, in the process of constructing the reduced form, only inner biderivations are needed in those steps where $\operatorname{rk}\psi_j > \operatorname{rk}\psi_i$. In particular, the corresponding equation $f_k(x) + b = 0$ appearing in the proof of Proposition 4.1 is linear over K. Consequently, the reduced form is defined over the original field K. ☐

The next definition will be used in Section 8.

Definition 4.4. Let $\Upsilon = \Upsilon(\underline{\psi}, \underline{\delta})$ be a triangular $\mathbf{t}$-module of dimension n, and let $\Upsilon^{\mathrm{red}} = \Upsilon(\underline{\psi}^{\mathrm{red}}, \underline{\delta}^{\mathrm{red}})$ denote its reduced form. We say that the triangular $\mathbf{t}$-module Υ allows for low-degree biderivations (abbreviated as LD-biderivations) if

$$\deg_\tau \delta_l^{\mathrm{red}} \ < \ \operatorname{rk}\psi_{l+1} \qquad \text{for } l = 1, 2, \ldots, n-1. \tag{4.6}$$

From the Corollary 4.2 we obtain the following:

Corollary 4.3. *Let $\Upsilon = \Upsilon(\underline{\psi}, \underline{\delta})$ be a triangular* $\mathbf{t}$*-module of dimension n defined over a field K, and suppose that*

$$\operatorname{rk}\psi_1 \ < \ \operatorname{rk}\psi_2 \ < \ \cdots \ < \ \operatorname{rk}\psi_n.$$

Then Υ allows for LD-biderivations and $K(\Upsilon) = K$.

Proof. Since Υ satisfies condition (4.5), we have $K(\Upsilon) = K$. Moreover,

$$\deg_\tau \delta_{i\times j}^{\mathrm{red}} \ < \ \max\{\operatorname{rk}\psi_i, \operatorname{rk}\psi_j\} = \operatorname{rk}\psi_j \qquad \text{for } i < j,$$

which proves the claim. ☐

It turns out that Corollary 4.3 admits a natural generalization to the case of non-strict inequalities, at the expense of a possible extension of the base field K. In this situation the reduction no longer requires any purely inseparable Frobenius extensions. Adjoining the relevant Artin–Schreier–Witt roots is sufficient. Hence, the reduced form arises after a purely separable base change of the type considered above, as stated in the following corollary.

Corollary 4.4. *Let $\Upsilon = \Upsilon(\underline{\psi}, \underline{\delta})$ be a triangular* $\mathbf{t}$*-module of dimension n defined over a field K, and suppose that*

$$\operatorname{rk}\psi_1 \ \leq \ \operatorname{rk}\psi_2 \ \leq \ \cdots \ \leq \ \operatorname{rk}\psi_n.$$

Then Υ allows for LD-biderivations, and its reduced form Υ^{red} is defined over an Artin–Schreier–Witt extension of K. $\square$

4.3. **Strict purity and almost strict purity.** Recall that a t-module Υ with

$$\Upsilon_t \;=\; \partial\Upsilon\,\tau^0 \;+\; \sum_{i=1}^{s} A_i\tau^i$$

is called *strictly pure* if the leading matrix A_s is invertible (cf. [NP24, Section 1.1]), and *almost strictly pure* if there exists an integer n such that the leading matrix of the polynomial Υ_{t^n} is invertible. Both strictly pure and almost strictly pure t-modules play a significant role in the arithmetic and analytic theory of t-modules. Almost strictly pure t-modules arise naturally in the study of periods, quasi-periods, and logarithms [L24, NP24], whereas strictly pure t-modules are precisely the class for which one can construct natural t-module structures on extension groups $\mathrm{Ext}^1_\tau(\Phi,\Psi)$ [KK24, GKK24], and for which duality theorems and Weil–Barsotti type formulas have been established [T95, PR03, KK26].

From the matrix expression (4.3), we therefore obtain the following characterization.

Proposition 4.5. *Let $\Upsilon = \Upsilon\big(\underline{\psi},\underline{\delta}\big)$ be a triangular module of dimension n. Then*

(*i*) Υ *is almost strictly pure iff*

$$\mathrm{rk}\,\psi_1 = \mathrm{rk}\,\psi_2 = \cdots = \mathrm{rk}\,\psi_n.$$

(*ii*) Υ *is strictly pure iff*

$$\deg_\tau \Upsilon = \mathrm{rk}\,\psi_1 = \mathrm{rk}\,\psi_2 = \cdots = \mathrm{rk}\,\psi_n.$$

$\square$

Moreover, Propositions 4.5 and 4.4 imply the following corollary.

Corollary 4.6. *Let $\Upsilon = \Upsilon(\underline{\psi},\underline{\delta})$ be a triangular, almost strictly pure* $\mathbf{t}$-*module, defined over a field K. Then there exists an Artin–Schreier–Witt extension $K(\Upsilon)/K$ such that Υ becomes isomorphic over $K(\Upsilon)$ to a strictly pure* $\mathbf{t}$*-module.*

Proof. Since Υ admits LD-biderivations, its reduced form Υ^{red} is a strictly pure $\mathbf{t}$-module. $\square$

5. Morphism between triangular t-modules

In this section we study morphism between triangular $\mathbf{t}$-modules. We begin with the following observation. Recall that there are no nonzero morphisms between Drinfeld modules of different ranks. In general, determining when Hom_τ vanishes in the category of $\mathbf{t}$-modules is a delicate problem. However, for triangular $\mathbf{t}$-modules we obtain the following generalization of the aforementioned fact:

Proposition 5.1. *Let* $\Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big)$ *and* $\widehat{\Upsilon} = \widehat{\Upsilon}\big(\underline{\widehat{\psi}}, \underline{\widehat{\delta}}\big)$ *be two triangular* $\mathbf{t}$*-modules. If* $\mathrm{rk}\,\psi_l \neq \mathrm{rk}\,\widehat{\psi}_j$ *for every* l, j *then*

$$\mathrm{Hom}_\tau(\Upsilon, \widehat{\Upsilon}) = \mathrm{Hom}_\tau(\widehat{\Upsilon}, \Upsilon) = 0.$$

Proof. Applying the functor $\mathrm{Hom}_\tau(\psi_l, -)$ consecutively to the sequences $\widehat{\delta}_j$ for $j = 1, 2, \ldots, \dim \widehat{\Upsilon} - 1$, we obtain that $\mathrm{Hom}_\tau(\psi_l, \widehat{\Upsilon}) = 0$ for every $l \in \{1, 2, \ldots, \dim \Upsilon\}$. Then we apply the functor $\mathrm{Hom}_\tau(-, \widehat{\Upsilon})$ consecutively to the sequences δ_l for $l = 1, 2, \ldots, \dim \Upsilon - 1$ and obtain that $\mathrm{Hom}_\tau(\Upsilon, \widehat{\Upsilon}) = 0$. The second equality follows by symmetry. □

5.1. **Triangular morphism.** Now we focus on morphisms given by triangular matrices. It turns out that they share a lot of analogous properties with the morphisms of Drinfeld modules.

Definition 5.1. Let $U : \Upsilon \longrightarrow \widehat{\Upsilon}$ be a morphism between two triangular $\mathbf{t}$-modules of the same dimension n. We say that U is a triangular morphism over K if $U \in \mathrm{Trian}_n\big(K\{\tau\}\big)$.

For a triangular morphism $U : \Upsilon \longrightarrow \widehat{\Upsilon}$, where $\dim \Upsilon = n$, we adopt the following enumeration of the entries of U.

$$U = \begin{bmatrix} u_n & 0 & \cdots & 0 & 0 \\ u_{n-1\times n} & u_{n-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{2\times n} & u_{2\times n-1} & \cdots & u_2 & 0 \\ u_{1\times n} & u_{1\times n-1} & \cdots & u_{1\times 2} & u_1 \end{bmatrix}. \tag{5.1}$$

Notice that this enumeration agrees with the convention (4.4) for triangular $\mathbf{t}$-modules

The notion of a triangular morphism is important since we have the following proposition:

Proposition 5.2. *Let* $V : \Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big) \longrightarrow \widehat{\Upsilon} = \widehat{\Upsilon}\big(\underline{\widehat{\psi}}, \underline{\widehat{\delta}}\big)$ *be a morphism defined over* K*. If* $\psi_i \not\sim \widehat{\psi}_j$*, i.e.* ψ_i *and* $\widehat{\psi}_j$ *are non-isogenous, for* $i \neq j$*, then* V *is triangular morphism.*

Proof. Let $V = (v_{i\times j}) \in M_{n,n}(K\{\tau\})$ be a matrix, with the usual indexing, such that

$$V \cdot \Upsilon_t = \widehat{\Upsilon}_t \cdot V.$$

Comparing the $1\times n$ terms of matrices in the above equality, we obtain $v_{1\times n}\cdot \phi_1 = \widehat{\phi}_n \cdot v_{1\times n}$. This means that $v_{1\times n}$ is an isogeny between ϕ_1 and $\widehat{\phi}_n$ and thus $v_{1\times n}=0$. Then we consider the $2\times n$ terms. We obtain the equality $v_{2\times n}\phi_1 = \widehat{\delta}_{n-1\times n}\cdot v_{1\times n} + \widehat{\phi}_{n-1}\cdot v_{2\times n}$. This in turn yields $v_{2\times n}=0$. Continuing, we see that $v_{i\times n}=0$ for $i=1,\dots,n-1$ and $v_{n\times n}$ is an isogeny between ϕ_1 and $\widehat{\phi}_1$. Then we compare the $1\times(n-1)$ terms and get $v_{1\times(n-1)}=0$ etc. So, going down (from the first to the diagonal term) in consecutive columns starting from the last we inductively obtain that $v_{i\times j}=0$ for $j>i$ and $v_{i\times i}$ is an isogeny between ϕ_{n-i+1} and $\widehat{\phi}_{n-i+1}$ for $i=n,\dots,1$. The fact that $v_{i\times i}$ is an isogeny is true for general triangular morphisms (cf. Prop. 5.4 (i)). $\square$

Corollary 5.3. *Let $\Upsilon = \Upsilon\big(\underline{\psi},\underline{\delta}\big)$ be a triangular $\mathbf{t}$-module such that $\psi_i \not\sim \psi_j$ for $i\neq j$. Then the endomorphism algebra* $\mathrm{End}_\tau(\Upsilon)$ *consists of triangular morphisms.* $\square$

We have the following proposition:

Proposition 5.4. *Let $U : \Upsilon = \Upsilon\big(\underline{\psi},\underline{\delta}\big) \longrightarrow \widehat{\Upsilon} = \widehat{\Upsilon}\big(\underline{\widehat{\psi}},\underline{\widehat{\delta}}\big)$ be a triangular morphism defined over K of the form* (5.1). *Then*

- (i) *The maps $u_i : \psi_i \longrightarrow \widehat{\psi}_i$ for $i=1,2,\dots,n$ are morphisms of Drinfeld modules.*
- (ii) *For the morphism U, the following conditions hold true*

  $$\widehat{\delta}_{j\times i}u_i - u_j\delta_{j\times i} + \sum_{k=j+1}^{i-1}\Big(\widehat{\delta}_{j\times k}u_{k\times i} - u_{j\times k}\delta_{k\times i}\Big) \in \mathrm{Der}_{in}\big(\psi_i,\widehat{\psi}_j\big)$$

  for every $j<i$.
  Moreover, U is determined uniquely by the family $\big\{u_i : \psi_i \longrightarrow \widehat{\psi}_i\big\}_{i=1}^{n}$ of morphisms of Drinfeld modules.
- (iii) *If* $\mathrm{char}_A K = 0$ *then* $\partial : \mathrm{Hom}_\tau(\Upsilon,\widehat{\Upsilon}) \longrightarrow \mathrm{Trian}_n(K)$ *is a monomorphism of $\mathbb{F}_q-$vector spaces.*

Proof. Comparing the diagonal entries in equation $U\Upsilon = \widehat{\Upsilon}U$, we obtain

$$u_i\psi_i = \widehat{\psi}_i u_i \quad \text{for each} \quad i=n,\dots,2,1.$$

Thus u_i are morphisms of Drinfeld modules, which proves (i).

For the proof of (ii) we again use the equality $U\Upsilon = \widehat{\Upsilon}U$. This time we compare the off diagonal terms at places $j\times i$ where $j<i$. We count the

rows starting from the bottom, and the columns going from right to left. Then for $j < i$ we obtain the following equalities:

$$u_{j\times i}\psi_i + \sum_{k=j+1}^{i-1} u_{j\times k}\delta_{k\times i} + u_j\delta_{j\times i} = \widehat{\delta}_{j\times i}u_i + \sum_{k=j+1}^{i-1} \widehat{\delta}_{j\times k}u_{k\times i} + \widehat{\psi}_j u_{j\times i}.$$

from which we get

$$\widehat{\delta}_{j\times i}u_i - u_j\delta_{j\times i} + \sum_{k=j+1}^{i-1}\left(\widehat{\delta}_{j\times k}u_{k\times i} - u_{j\times k}\delta_{k\times i}\right) = \\ = u_{j\times i}\psi_i - \widehat{\psi}_j u_{j\times i} \in \mathrm{Der}_{in}\big(\psi_i, \widehat{\psi}_j\big). \tag{5.2}$$

Now, we use equation (5.2) for the following pairs $(i-1, i)$ where $i = n, n-1, \ldots, 2$, in order to obtain the terms in the first sub-diagonal of U. For example, from the condition $\widehat{\delta}_{n-1\times n}u_n - u_{n-1}\delta_{n-1\times n} \in \mathrm{Der}_{in}\big(\psi_n, \widehat{\psi}_{n-1}\big)$, we see that there exists a unique $u_{n-1\times n}$ such that

$$\widehat{\delta}_{n-1\times n}u_n - u_{n-1}\delta_{n-1\times n} = u_{n-1\times n}\psi_n - \widehat{\psi}_{n-1}u_{n-1\times n}.$$

Thus, the proof of (ii) is completed by induction on terms on consecutive sub-diagonals.

The fact that ∂ is a linear map follows immediately from the definition. It remains to be shown that δ is a monomorphism. Let $U \in \ker\partial$. Then

$$\partial U = \begin{bmatrix} \partial u_n & 0 & \cdots & 0 & 0 \\ \partial u_{n-1\times n} & \partial u_{n-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \partial u_{2\times n} & \partial u_{2\times n-1} & \cdots & \partial u_2 & 0 \\ \partial u_{1\times n} & \partial u_{1\times n-1} & \cdots & \partial u_{1\times 2} & \partial u_1 \end{bmatrix} = 0. \tag{5.3}$$

Since $\partial u_i = 0$ and u_i is a morphism of Drinfeld modules, $u_i = 0$ for $i = n, \ldots, 2, 1$. This shows that all entries on the diagonal of U are zero. Now consider the equality (5.2) for the pair $(j, i) = (l-1, l)$. We have

$$\widehat{\delta}_{l-1\times l}u_l - u_{l-1}\delta_{l-1\times l} = u_{l-1\times l}\psi_l - \widehat{\psi}_{l-1}u_{l-1\times l},$$

where the left-hand side vanishes, since u_l and u_{l-1} are zero maps. This shows that $u_{l-1\times l}$ is a morphism of Drinfeld modules for which $\partial u_{l-1\times l} = 0$. Therefore, $u_{l-1\times l} = 0$. So, all entries at the sub-diagonal $(l-1, l)$ are zero. Consider the equality (5.2) for the pairs $(j, i) = (l-2, l)$. Then we have

$$\widehat{\delta}_{l-2\times l}u_l - u_{l-2}\delta_{l-2\times l} + \left(\widehat{\delta}_{l-2\times l-1}u_{l-1\times l} - u_{l-1\times l}\delta_{l-1\times l}\right) = u_{l-2\times l}\psi_l - \widehat{\psi}_{l-2}u_{l-2\times l}.$$

Notice again that the left-hand side is zero. Therefore, $u_{l-2\times l}$ is a morphism of Drinfeld modules for which $\partial u_{l-2\times l} = 0$. By finite induction we obtain $U = 0$. $\square$

Recall that the space $\mathrm{Hom}_\tau(\Upsilon, \widehat{\Upsilon})$ is an $\mathbb{F}_q[t]-$module with the multiplication by $a \in \mathbb{F}_q[t]$ defined as

$$a \star U := U \cdot \Upsilon_a.$$

Analogously, we can define an $\mathbb{F}_q[t]-$module structure on $\mathrm{Trian}_n(K)$. Then ∂ is a morphism of $\mathbb{F}_q[t]-$modules.

5.2. **Triangular isogenies.** Recall also that a surjective morphism of **t**-modules with a finite kernel is called an isogeny, see [H19, Definition 5.1.]. The following proposition gives a characterization of triangular isogenies.

Proposition 5.5. *Let* $U : \Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big) \longrightarrow \widehat{\Upsilon} = \widehat{\Upsilon}\big(\underline{\widehat{\psi}}, \underline{\widehat{\delta}}\big)$ *be a triangular morphism defined over* K^{sep}*, of the form* (5.1) *. Then* U *is an isogeny iff* $u_i : \psi_i \longrightarrow \widehat{\psi}_i$ *are isogenies of Drinfeld modules for all* $i = 1, 2, \ldots, n$*.*

Proof. Let U be an isogeny. If for some i we have $u_i = 0$, then one readily sees that the kernel of U cannot be finite since for a standard unit vector e_i, i.e. $(e_i)_j = \delta_{ij}$, the subspace $X = K^{\mathrm{sep}} e_i \subset \ker U$ and K^{sep} is infinite.

Conversely, let U be a triangular morphism of the form (5.1) such that $u_i : \psi_i \longrightarrow \widehat{\psi}_i$ is an isogeny for every i.

First, we will show that U has finite kernel. Assume $x = (x_n, x_{n-1}, \ldots, x_2, x_1)^T \in \ker U$. The equation $Ux = 0$ yields the following system of equations:

$$\begin{aligned}
&u_n x_n = 0\\
&u_{n-1\times n} x_n + u_{n-1} x_{n-1} = 0\\
&\vdots\\
&\sum_{i=2}^{n} u_{1\times i} x_i + u_1 x_1 = 0.
\end{aligned}$$

From the first equation, we see that x_n is a root of the nonzero polynomial u_n. So, there are only a finite number of possible values for x_n. For fixed x_n the second equation has finitely many solutions x_{n-1}. By induction it follows that $\ker U$ is finite.

Now, we will show that U is surjective. Let $y = (y_n, \ldots, y_2, y_1)^T \in K^n$. The system $Ux = y$ gives the following system of n equations:

$$\begin{aligned}
&u_n x_n = y_n \\
&u_{n-1\times n} x_n + u_{n-1} x_{n-1} = y_{n-1} \\
&\vdots \\
&\sum_{i=2}^{n} u_{1\times i} x_i + u_1 x_1 = y_1,
\end{aligned}$$

which again can be solved starting from the first equation then passing to the second etc. This shows that U is surjective with a finite kernel, i.e. an isogeny.

□

A triangular morphism, which is an isogeny will be called a triangular isogeny.

For a Drinfeld module ψ let $H(\psi)$ be its height (see [P23, Lemma 3.2.11].

Then from [P23, Proposition 3.3.4] we obtain the following corollary.

Corollary 5.6. *If $U : \Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big) \longrightarrow \widehat{\Upsilon} = \widehat{\Upsilon}\big(\underline{\widehat{\psi}}, \underline{\widehat{\delta}}\big)$ is a triangular isogeny then $H(\psi_i) = H(\widehat{\psi}_i)$ for all i.* □

According to [H19, Corollary 5.15.] for every isogeny of $\mathbf{t}$-modules there exists a dual isogeny satisfying:

$$U \circ U^{\vee} = \widehat{\Upsilon}_a \quad \text{and} \quad U^{\vee} \circ U = \Upsilon_a, \quad \text{for some} \quad a \in \mathbb{F}_q[t]. \tag{5.4}$$

So, we obtain the following corollary:

Corollary 5.7. *For every triangular isogeny there exists a dual isogeny which is triangular.* □

5.3. **Endomorphism algebra of triangular morphisms.** Recall that in characteristic zero the endomorphism algebra of a Drinfeld module is commutative. An analogous result holds true for triangular $\mathbf{t}$-modules with non-isogenous sub-quotients.

Proposition 5.8. *Let $\mathrm{char}_A K = 0$ and $\Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big)$ be a triangular $\mathbf{t}$-module of dimension n with pairwise non-isogenous sub-quotients i.e. $\psi_i \not\sim \psi_j$ for $i \neq j$. Then the endomorphism algebra $\mathrm{End}_\tau(\Upsilon)$ is commutative.*

Proof. By Corollary 5.3, $\mathrm{End}_\tau(\Upsilon)$ consists of triangular endomorphisms. So, consider two triangular morphisms $W, V \in \mathrm{End}_\tau(\Upsilon)$. Then the commutator $WV - VW$ is again a triangular endomorphism of the triangular $\mathbf{t}$-module

Υ. Denote it by U, and its entries according to (5.1). Since u_i for $i = 1, \ldots, n$ is a commutator of endomorphisms of the Drinfeld module ψ_i, we get $u_i = 0$ for $i = 1, \ldots, n$. Similarly as in the proof of Proposition 5.4, putting in the equality (5.2) $i = l$ and $j = l-1$, we obtain that $u_{l-1\times l} \in \mathrm{Hom}_\tau(\psi_l, \psi_{l-1}) = 0$. Therefore $u_{l-1\times l} = 0$ for all l. Substituting $i = l$ and $j = l-2$ into equality (5.2), we obtain that $u_{l\times l-2} \in \mathrm{Hom}_\tau(\psi_l, \psi_{l-2}) = 0$. By a finite induction we obtain that $U = 0$.

$\square$

Proposition 5.9. *Let* $\mathrm{char}_A K = 0$ *and* $\Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big)$ *be defined over* K *such that* $\mathrm{rk}\,\psi_i \neq \mathrm{rk}\,\psi_j$ *for* $i \neq j$. *Then*

- *(i) For each* $U_0 \in \mathrm{Trian}_n\big(K^{\mathrm{sep}}\big)$, *one can effectively decide whether there exists* $U \in \mathrm{End}_{K^{\mathrm{sep}}}(\Upsilon)$ *with* $\partial U = U_0$, *and, if so, find it.*
- *(ii) If* $U \in \mathrm{End}_{K^{\mathrm{sep}}}(\Upsilon)$ *has* $\partial U \in \mathrm{Trian}_n\big(K\big)$, *then* $U \in \mathrm{End}_K(\Upsilon)$.

For Drinfeld modules, a corresponding result was proved by Kuhn and Pink in [KP22, Proposition 5.3]. Before we give the proof of Proposition 5.9, we need the following lemma:

Lemma 5.10. *Let*

$$\psi_t \cdot c(\tau) - c(\tau) \cdot \phi_t = w(\tau) \tag{5.5}$$

be an equation, where ψ, ϕ *are fixed Drinfeld modules defined over* $L \supset K$ *of characteristic zero and* $w(\tau)$ *is a fixed skew polynomial defined over* L. *Then*

- *(i) if* $\partial w(\tau) \neq 0$, *then the equation* (5.5) *has no solution.*
- *(ii) If* $\partial w(\tau) = 0$, *then every solution* $c(\tau)$ *of the equation* (5.5) *fulfills the following conditions:*
  - *(1)* $\deg_\tau c(\tau) \geq \deg_\tau w(\tau) - \max\{\mathrm{rk}\,\psi, \mathrm{rk}\,\phi\}$
  - *(2)* $c(\tau)$ *is uniquely determined by* $\partial c(\tau)$, *i.e. if* $c(\tau) = \sum\limits_{i=0}^{r} c_i\tau^i$, *then* $c_i = f_i(c_0)$ *for some skew polynomials* $f_i(\tau) \in L\{\tau\}$ *that depend on* ψ, ϕ *and* $w(\tau)$.
  - *(3)* c_0 *is a root of the system of exactly* $\max\{\mathrm{rk}\,\psi, \mathrm{rk}\,\phi\}$ *polynomials that depend on* ϕ, ψ, $w(\tau)$ *and* $\deg_\tau c(\tau)$.
  - *(4) additionally, if* $\mathrm{rk}\,\psi \neq \mathrm{rk}\,\phi$ *then* $\deg_\tau c(\tau) = \deg_\tau w(\tau) - \max\{\mathrm{rk}\,\psi, \mathrm{rk}\,\phi\}$.

Proof. Let $\psi = \theta + \sum\limits_{i=1}^{m} \alpha_i\tau^i$, $\phi = \theta + \sum\limits_{i=1}^{n} \beta_i\tau^i$ and $w(\tau) = \sum\limits_{i=0}^{w} w_i\tau^i$. Denote $M = \max\{n, m\}$ and consider a solution of the following form: $c(\tau) =$

$\sum_{i=0}^{r} c_i\tau^i$. Then the equality (5.5) can be rewritten as follows:

$$\sum_{k=1}^{r}\left(\left(\theta-\theta^{(k)}\right)c_k+\sum_{i=i_k}^{k-1}\left(\alpha_{k-i}c_i^{(k-i)}-c_i\beta_{k-i}^{(i)}\right)\right)\tau^k+ \\ +\sum_{k=r+1}^{r+M}\left(\sum_{i=i_k}^{k-1}\left(\alpha_{k-i}c_i^{(k-i)}-c_i\beta_{k-i}^{(i)}\right)\right)\tau^k=\sum_{k=0}^{w}w_k\tau^k,$$

where $i_k=\max\{0,k-M\}$. We put $\alpha_j=0$ for $j>m$ and $\beta_l=0$ for $l>n$.

Comparing the coefficients at the least power we obtain (i). Comparison of the degrees yields a contradiction if $r<w-M$, which proves (1). Comparing the coefficients at τ^k for $k=1,2,\ldots,r$ we obtain that

$$c_k=\frac{1}{\theta-\theta^{(k)}}\cdot\sum_{i=i_k}^{k-1}\left(c_i\beta_{k-i}^{(i)}-\alpha_{k-i}c_i^{(k-i)}\right)+\frac{w_k}{\theta-\theta^{(k)}}\quad\text{for}\quad k=1,2,\ldots,r,$$

where we put $w_*=0$ for $*>w$. This allows one to determine consecutively c_1, c_2, ..., c_r as polynomial functions of c_0. This proves (2).

Comparing the coefficients at τ^k for $k=r+1,r+2,\ldots r+M$ we obtain the following equalities:

$$\sum_{i=i_k}^{k-1}\left(\alpha_{k-i}c_i^{(k-i)}-c_i\beta_{k-i}^{(i)}\right)=w_k\quad\text{for}\quad k=r+1,r+2,\ldots,r+M.$$

Substituting $c_i=f_i(c_0)$ we have

$$g_k(c_0):=w_k-\sum_{i=i_k}^{k-1}\left(\alpha_{k-i}f_i^{(k-i)}(c_0)-f_i(c_0)\beta_{k-i}^{(i)}\right)=0\quad\text{for}\quad k=r+1,\ldots,r+M.$$

Therefore c_0 is a root of the system of polynomials $g_{r+1},g_{r+2},\ldots,g_{r+M}$. This shows (3).

For the proof of (4) assume that $m>n$. (The case $n<m$ is proved analogously.) Then the equation (5.5) can be rewritten in the following form:

$$\psi_t\cdot c(\tau)-w(\tau)=c(\tau)\cdot\phi_t.$$

If $r\geq w-M$ then comparison of the degrees in the above equation implies that $r=w-M$. □

Proof of the Proposition 5.9. Assume we have a matrix $U_0\in\mathrm{Trian}_n\left(K^{\mathrm{sep}}\right)$. We would like to decide whether there exists a matrix U of the form (5.1) such that $\partial U=U_0$. By [KP22, Proposition 5.3] we can effectively decide whether there exist morphisms $u_i\in\mathrm{End}(\psi_i),\ i=1,\ldots,n$ such that $\partial u_i=U_0(i,i)$. The diagonal term of U_0, and if so then effectively find them. If such morphisms exist then we apply consecutively Lemma 5.10. First, to the pairs (i,j) of the form $(l,l-1)$, $l=n,\dots,2$. Then $(l,l-2)$, $l=n,\dots,3$ etc. We finish the finite induction once we reach the pair $(n,1)$. Notice that the above quoted nontrivial result of N. Kuhn and R. Pink was needed to have control over the τ-degree of the right-hand side of the equation (5.5) at the initial step of induction, and point (4) of Lemma 5.10 was essential for providing upper bounds for the degrees of the right-hand side of (5.5) at every step of induction. □

Remark 5.1. Notice that in the proof of Proposition 5.9 the assumption concerning different ranks of subquotients ψ_i was necessary for bounding the degree of $c(\tau)$ in every inductive step . It would be nice to have a bound for degree of $c(\tau)$ in arbitrary situations, see Lemma 5.10 point (3).

6. Exponent for an exact sequence

In this section we consider analytical properties of triangular $\mathbf{t}$-modules. We start with the following lemma.

Lemma 6.1. *Let $Y\in \mathrm{Mat}_{n\times m}(K)$ be an arbitrary matrix and $M\in \mathrm{Mat}_m(K)$, $N\in \mathrm{Mat}_n(K)$ be lower triangular matrices. Then there exists a unique matrix $C\in \mathrm{Mat}_{n\times m}(K)$ which fulfills the following equation*

$$C+CM+NC=Y. \tag{6.1}$$

Proof. We determine the entries of the matrix C row by row, starting from the top row and proceeding downward. Within each row, the entries are computed from right to left. □

From this point until the end of this section we work over the field $K=\mathcal{C}_\infty$. The following theorem describes the behavior of the exponential function associated with the middle term of a short exact sequence. Let us note that this fact, in the case where $\partial\delta=0$, was already established in [PR03].

Theorem 6.2. *Let $0\longrightarrow \Phi\longrightarrow \Upsilon\longrightarrow \Psi\longrightarrow 0$ be an exact sequence of $\mathbf{t}$-modules given by the biderivation δ. Then there exists the following commutative diagram of $\mathbb{F}_q[t]-$modules with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & \Phi \arrow[r, "{\begin{bmatrix}0\\1\end{bmatrix}}"] \arrow[d, "\mathrm{Exp}_\Phi"] & \Upsilon \arrow[r, "{\begin{bmatrix}1 & 0\end{bmatrix}}"] \arrow[d, "\mathrm{Exp}_\Upsilon"] & \Psi \arrow[r] \arrow[d, "\mathrm{Exp}_\Psi"] & 0 \\
0 \arrow[r] & {}^{\Phi}K^d \arrow[r, "{\begin{bmatrix}0\\1\end{bmatrix}}"'] & {}^{\Upsilon}K^{d+e} \arrow[r, "{\begin{bmatrix}1 & 0\end{bmatrix}}"'] & {}^{\Psi}K^e \arrow[r] & 0,
\end{tikzcd} \tag{6.2}$$

where Exp_Φ*, resp.* Exp_Ψ*, are exponents for* Φ*, resp.* Ψ*. The exponent of* Υ *has the following form:*

$$\mathrm{Exp}_\Upsilon(\tau) = \left[\begin{array}{c|c} \mathrm{Exp}_\Psi(\tau) & 0 \\ \hline \mathrm{Exp}_\delta(\tau) & \mathrm{Exp}_\Psi(\tau), \end{array}\right], \tag{6.3}$$

where $\mathrm{Exp}_\delta(\tau)$ *is uniquely determined by the following equation:*

$$\mathrm{Exp}_\delta\big(\partial\Psi_t\tau^0\big) - \Phi_t\mathrm{Exp}_\delta(\tau) = \delta_t\mathrm{Exp}_\Psi(\tau) - \mathrm{Exp}_\Phi\big(\partial\delta_t\tau^0\big). \tag{6.4}$$

Proof. Notice that if we show that $\mathrm{Exp}_\Upsilon(\tau)$ is of the form (6.3), then commutativity of the diagram (6.2) is obvious.

Since $\mathrm{Exp}_\Upsilon(\tau) = I\tau^0 + \sum\limits_{i\geq 1} C_i\tau^i$ fulfills the following equation: $\mathrm{Exp}_\Upsilon(\partial\Upsilon_t\tau^0) = \Upsilon_t\mathrm{Exp}_\Upsilon(\tau)$ we obtain the following equality of the formal power series:

$$\begin{aligned}\sum_{i\geq 1}\Big(C_i\partial\Upsilon_t^{(i)} - \partial\Upsilon_t C_i\Big)\tau^i = \sum_{i=1}^{s} U_i\tau^i + \sum_{i\geq 1} U_1 C_i^{(1)}\tau^{i+1} + \\ + \sum_{i\geq 1} U_2 C_i^{(2)}\tau^{i+2} + \cdots + \sum_{i\geq 1} U_s C_i^{(s)}\tau^{i+s}\end{aligned}$$

where $\Upsilon_t = \partial\Upsilon_t\tau^0 + \sum\limits_{i=1}^{s} U_i\tau^i$. Comparing coefficients at τ^k, we obtain

$$C_k\partial\Upsilon_t^{(k)} - \partial\Upsilon_t C_k = U_k + \sum_{i=1}^{k-1} U_i C_{k-i}^{(i)} \qquad \text{for} \quad k\leq s \tag{6.5}$$

$$C_k\partial\Upsilon_t^{(k)} - \partial\Upsilon_t C_k = \sum_{i=1}^{s} U_i C_{k-i}^{(i)} \qquad \text{for} \quad k > s \tag{6.6}$$

Let $\Phi_t = (\theta I + N_\Phi)\tau^0 + \sum\limits_{i=1}^{\text{finite}} A_i\tau^i$, $\Psi_t = (\theta I + N_\Psi)\tau^0 + \sum\limits_{i=1}^{\text{finite}} B_i\tau^i$ and $\delta_t = \sum\limits_{i=0}^{\text{finite}} D_i\tau^i$. Without loss of generality we can assume that N_Φ and N_Ψ are lower triangular matrices. Then

$$\Upsilon_t = \left[\begin{array}{c|c} \Psi_t & 0 \\ \hline \delta_t & \Psi_t \end{array}\right] = \left[\begin{array}{c|c} \theta I + N_\Psi & 0 \\ \hline D_0 & \theta I + N_\Phi \end{array}\right]\tau^0 + \sum_{i=1}^{s}\left[\begin{array}{c|c} B_i & 0 \\ \hline D_i & A_i \end{array}\right]\tau^i.$$

Putting $C_i = \left[\begin{array}{c|c} C_{1,i} & C_{2,i} \\ \hline C_{3,i} & C_{4,i} \end{array}\right]$ in the equalities (6.5) and (6.6), we obtain $C_{2,i} = 0$ for $i \geq 1$. Indeed, comparing the blocks in the upper right corners we obtain the following relation:

$$C_{2,1}\Big(\theta^{(1)}I + N_\Phi^{(1)}\Big) - \Big(\theta I + N_\Psi\Big)C_{2,1} = 0,$$

which can be rewritten in the following form:

$$\Big(\theta^{(1)} - \theta\Big)C_{2,1} + C_{2,1}N_\Phi^{(1)} - N_\Psi C_{2,1} = 0.$$

Dividing both sides by $\theta^{(1)} - \theta$, we obtain (6.1), which by lemma 6.1 has the unique solution $C_{2,1} = 0$. Assume that the subsequent matrices $C_{2,i}$ for $i = 1, 2, \cdots k-1$ are zero matrices. We will show the k-th matrix is zero. Since the arguments for both $k \le s$ and $k > s$ are analogous, we will consider only the first case. In the formula (6.5) all block matrices U_i and $C^{(i)}_{k-i}$ are lower triangular. Therefore the matrix $C_k \partial \Upsilon^{(k)}_t - \partial \Upsilon_t C_k$ is also block lower triangular. Thus, we have the following equality

$$C_{2,k}\Big(\theta^{(k)} I + N^{(k)}_{\Phi}\Big) - \Big(\theta I + N_{\Psi}\Big) C_{2,k} = 0,$$

which by lemma 6.1 has the only solution $C_{2,k} = 0$.

We obtained the following formula:

$$\mathrm{Exp}_{\Upsilon} = I\tau^0 + \sum_{i\ge 1} \left[\begin{array}{c|c} C_{1,i} & 0 \\ \hline C_{3,i} & C_{4,i} \end{array}\right] \tau^i = \left[\begin{array}{c|c} I\tau^0 + \sum\limits_{i\ge 1} C_{1,i}\tau^i & 0 \\ \hline \sum\limits_{i\ge 1} C_{3,i}\tau^i & I\tau^0 + \sum\limits_{i\ge 1} C_{4,i}\tau^i \end{array}\right].$$

Substituting this into equality $\mathrm{Exp}_{\Upsilon}(\partial \Upsilon_t \tau^0) = \Upsilon_t \mathrm{Exp}_{\Upsilon}(\tau)$ and comparing the block matrices on the diagonal, we obtain the following two conditions:

$$\Big(I\tau^0 + \sum_{i\ge 1} C_{1,i}\tau^i\Big)\partial \Psi_t \tau^0 = \Psi_t \Big(I\tau^0 + \sum_{i\ge 1} C_{1,i}\tau^i\Big)$$
$$\Big(I\tau^0 + \sum_{i\ge 1} C_{4,i}\tau^i\Big)\partial \Phi_t \tau^0 = \Phi_t \Big(I\tau^0 + \sum_{i\ge 1} C_{4,i}\tau^i\Big).$$

The first implies that $I\tau^0 + \sum\limits_{i\ge 1} C_{1,i}\tau^i = e_{\Psi}(\tau)$, whereas the second implies that $I\tau^0 + \sum\limits_{i\ge 1} C_{4,i}\tau^i = \mathrm{Exp}_{\Phi}(\tau)$. Finally, comparing the matrices in the lower left corner and denoting $\mathrm{Exp}_{\delta}(\tau) = \sum\limits_{i\ge 1} C_{3,i}\tau^i$ we obtain the following equation

$$\mathrm{Exp}_{\delta}\Big(\partial \Psi_t \tau^0\Big) + \mathrm{Exp}_{\Phi}\Big(\partial \delta_t \tau^0\Big) = \delta_t \mathrm{Exp}_{\Psi}(\tau) + \Phi_t \mathrm{Exp}_{\delta}(\tau),$$

which is equivalent to (6.4).

To finish the proof one has to show that the equality (6.4) determines $\mathrm{Exp}_{\delta}(\tau)$ uniquely.

In order to do this, notice that the right-hand side of the aforementioned equality is of the following form: $\sum\limits_{i\ge 1} Y_i \tau^i$ for certain matrices Y_i. Then putting $\mathrm{Exp}_{\delta}(\tau) = \sum\limits_{i\ge 1} E_i \tau^i$ and comparing the coefficients at τ^k we obtain the following equalities

$$E_k\Big(\theta I + N_{\Psi}\Big)^{(k)} - \Big(\theta I + N_{\Phi}\Big) E_k = Y_k + \sum_{i=1}^{m} A_i E^{(i)}_{k-i},$$

where $m = \min\{k-1, \deg \Phi_t\}$. Assuming inductively that the matrices E_1, $E_2, \dots, E_{k-1}$ have already been determined, one can find E_k by transforming the above equality to the form (6.1) and applying lemma 6.1. □

As a consequence of this theorem, we obtain the following interesting fact about uniformizable $\mathbf{t}$-modules.

Corollary 6.3. *The class of uniformizable* $\mathbf{t}$*-modules is closed with respect to extensions and images.*

Proof. Applying the snake lemma to the diagram (6.2) we obtain the following exact sequence:

$$\begin{aligned} (6.7) \qquad & 0 \longrightarrow \ker \mathrm{Exp}_{\Phi} \longrightarrow \ker \mathrm{Exp}_{\Upsilon} \longrightarrow \ker \mathrm{Exp}_{\Psi} \longrightarrow \\ & \longrightarrow \operatorname{coker} \mathrm{Exp}_{\Phi} \longrightarrow \operatorname{coker} \mathrm{Exp}_{\Upsilon} \longrightarrow \operatorname{coker} \mathrm{Exp}_{\Psi} \longrightarrow 0, \end{aligned}$$

from which the corollary follows immediately. □

Moreover, in the case of uniformizable $\mathbf{t}$-modules we obtain the following description of the lattice corresponding to the middle $\mathbf{t}$-module in the short exact sequence.

Corollary 6.4. *If in a short exact sequence* $0 \longrightarrow \Phi \longrightarrow \Upsilon \longrightarrow \Psi \longrightarrow 0$*, a* $\mathbf{t}$*-module* Φ *is uniformizable, then the lattice* Λ_{Υ} *as an* $\mathbb{F}_q[t]-$*module is isomorphic to the product of lattices* $\Lambda_{\Phi} \times \Lambda_{\Psi}$.

Proof. From (6.7) we obtain the following short exact sequence:

$$0 \longrightarrow \ker \mathrm{Exp}_{\Phi} \longrightarrow \ker \mathrm{Exp}_{\Upsilon} \longrightarrow \ker \mathrm{Exp}_{\Psi} \longrightarrow 0.$$

Since $\ker \mathrm{Exp}_{\Psi} = \Lambda_{\Psi}$ is a projective $\mathbb{F}_q[t]-$module this sequence splits. Thus, $\Lambda_{\Upsilon} = \ker \Upsilon \cong \ker \mathrm{Exp}_{\Phi} \times \ker \mathrm{Exp}_{\Psi} = \Lambda_{\Phi} \times \Lambda_{\Psi}$, as $\mathbb{F}_q[t]-$modules. □

Next, we may extend the last two corollaries by induction to τ-composition series of $\mathbf{t}$-modules.

Corollary 6.5. *Let*

$$\Upsilon_1 \hookrightarrow \Upsilon_2 \hookrightarrow \cdots \hookrightarrow \Upsilon_n = \Upsilon,$$

be a $\tau-$*composition series where the quotients* $\Psi_i = \Upsilon_{i+1}/\Upsilon_i$ *are uniformizable* $\mathbf{t}$*-modules for* $i = 0, 1, \dots, n-1$*. Then*

(*i*) Υ *is a uniformizable* $\mathbf{t}$*-module.*
(*ii*) $\Lambda_{\Upsilon} \cong \prod_{i=0}^{n-1} \Lambda_{\Psi_i}$ *as* $\mathbb{F}_q[t]-$*modules.*

Proof. Consider the following exact sequences:

$$\eta_i : \quad 0 \longrightarrow \Upsilon_i \longrightarrow \Upsilon_{i+1} \longrightarrow \Psi_i \longrightarrow 0$$

for $i = 1, 2, \ldots, n-1$. Applying the last two corollaries successively to sequences $\eta_1, \eta_2, \ldots, \eta_{n-1}$ we obtain the assertion. $\square$

Corollary 6.6. *Every triangular* $\mathbf{t}$*-module is uniformizable.* $\square$

Corollary 6.7. *Triangular* $\mathbf{t}$*-module* $\Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big)$ *has rank* $r = \sum_i \operatorname{rk} \psi_i$.

Proof. Let $r_i = \operatorname{rk} \psi_i$ and λ_{i,k_i}, $k_i = 1, \ldots, \operatorname{rk} \psi_i$ be the $(\partial\psi_i)(A)$ generators of Λ_{ψ_i}. Applying an easy induction using Corollary 6.4 one sees that the $(\partial\Upsilon)(A)$-basis of Λ_Υ is given by

$$\big[\lambda_{n,1}, \alpha_{n,1,2}, \ldots, \alpha_{n,1,n}\big]^T, \ldots, \big[\lambda_{n,r_n}, \alpha_{n,r_n,2}, \ldots, \alpha_{n,r_n,n}\big]^T,$$
$$\big[0, \lambda_{n-1,1}, \alpha_{n-1,1,3} \ldots, \alpha_{n-1,1,n}\big]^T, \ldots, \big[0, \ldots, 0, \lambda_{1,k_1}\big]^T$$

for some $\alpha_{s,u,v} \in \mathbb{C}_\infty$. The corollary follows. $\square$

7. Finiteness and purity

In this section, we will show that triangular modules are finite and pure. As a consequence, we will obtain conclusions concerning Taelman's conjecture.

7.1. **Proofs of finiteness and purity.** In this section, we prove that triangular modules are finite. In order to do this, we will use the results of A. Maurischat [M21]. The citerion for finiteness developed there uses the Smith normal form of a matrix over a suitable skew field. We start with a general setting. It is well-known that any matrix $M \in \operatorname{Mat}_{n\times m}(K[x])$, over a polynomial ring $K[x]$, where K is a field, can be reduced by means of elementary operations to the diagonal form, i.e. an equivalent matrix M^{red} has the property $M^{\mathrm{red}}_{i,i} = d_i, \quad i = 1, \ldots, \min(m,n)$ and $M^{\mathrm{red}}_{i,j} = 0$ otherwise. The elementary divisors $d_i(x)$ are monic polynomials, satisfying the following divisibility property $d_1|d_2|\ldots|d_{\min(m,n)}$.

By elementary operations, we mean operations of the following types:

(i) the interchange of two rows or two columns
(ii) changing the sign of a row or a column,
(iii) addition of a polynomial multiple of one row/column to a different row/column.

The algorithm for reducing a matrix M to its Smith normal form is a consequence of the Euclidean algorithm.

Now, consider the twisted polynomial ring $K\{\tau\}$, i.e. a polynomial ring in the variable τ over a field K with the commutativity relation $\tau\cdot\alpha = \alpha^q\cdot\tau$. Assume K is perfect. Let $K\{\{\sigma\}\} = \big\{\sum_{i=0}^{\infty}\alpha_i\sigma^i, \quad \alpha_i \in K\big\}$, where $\sigma = \tau^{-1}$, be the ring of power series in σ with the coefficients in K. Then we have $\sigma \cdot \alpha = \alpha^{1/q} \cdot \sigma, \quad \alpha \in K$ and the ring of the Laurent series:

$$K(\{\sigma\}) = \Big\{ \sum_{i=i_0}^{\infty} \alpha_i \sigma^i, \quad \alpha_i \in K \Big\} \tag{7.1}$$

is a skew field. The ring $K\{\tau\}$ can be naturally embedded in $K\{\sigma\}$ by the formula:

$$\sum_{i=0}^{n} \alpha_i \tau^i \to \sum_{i=0}^{n} \alpha_i \sigma^{-i} \tag{7.2}$$

Notice that in the polynomial ring $K(\{\sigma\})[t]$ there exist both a left and right division algorithm. Therefore, any matrix $M \in \mathrm{Mat}_{n\times m}(K(\{\sigma\})[t])$ can be reduced via elementary operations to the Smith normal form, i.e. to the diagonal matrix with $\lambda_1|\lambda_2 \ldots |\lambda_d$. Here λ_i divides λ_{i+1} from both left and right sides.

Let Υ be a $\mathbf{t}$-module of dimension d and let $\Upsilon_t \in \mathrm{Mat}_{d\times d}(K\{\tau\})$ be the matrix corresponding to the multiplication by t. Then we can consider Υ_t as a matrix in $\mathrm{Mat}_{d\times d}(K(\{\sigma\}))$ via the embedding (7.2). Let λ_i, where $i = 1, \ldots, d$ be the invariant factors of Υ_t obtained by diagonalizing the matrix $tI_d - \Upsilon_t$. We also have a definition analogous to the commutative case .

Definition 7.1. A Newton polygon of a polynomial $p(t) = \sum_{i=0}^{n} a_i t^i \in K(\{\sigma\})[t]$ is the lower convex hull of the set of points $P_i = (i, \mathrm{ord}_\sigma(a_i)), \quad i = 0, \ldots, n$.

In the non-commutative case, the invariant factors λ_i are not unique but the orders in σ of their coefficients are unique once we assume that λ_i's are monic. This can be achieved as a consequence of the following theorem (cf. [C95, p.380]).

Theorem 7.1. [M21, Theorem 3.6] *Let $\mathcal{L}$ be a complete discretely valued skew field. Given a left $\mathcal{L}[t]$-module M, which is finitely generated as $\mathcal{L}$-vector space of dimension d, there exist monic polynomials $\lambda_1, \lambda_2, \ldots, \lambda_d \in \mathcal{L}[t] \setminus \{0\}$ such that λ_{i+1} is both left-divisible and right-divisible by λ_i for all $i = 1, \ldots, d-1$, and*

$$M \cong \mathcal{L}[t]/\lambda_1\mathcal{L}[t] \oplus \cdots \oplus \mathcal{L}[t]/\lambda_d\mathcal{L}[t]$$

as $\mathcal{L}[t]$-modules.

Additionally, one has the following result:

Theorem 7.2. [M21, Theorem 3.8] *Let $\mathcal{L}$ be as in the previous theorem. Given a left $\mathcal{L}[t]$-module M, which is finitely generated as $\mathcal{L}$-vector space of dimension d, there exist monic polynomials $f_1, \dots, f_k \in \mathcal{L}[t] \setminus \mathcal{L}, \quad k \leq d$ such that the Newton polygon of each f_i consists of one edge, and*

$$M \cong \mathcal{L}[t]/f_1\mathcal{L}[t] \oplus \cdots \oplus \mathcal{L}[t]/f_k\mathcal{L}[t]$$

as $\mathcal{L}[t]$-modules.

In this setting, A. Maurischat proved the following remarkable results:

Theorem 7.3. [M21, Theorem 6.6] *For the $\mathbf{t}$-module (E, Υ_t) with Υ_t being represented by the matrix D the following are equivalent*

(1) E is abelian
(2) E is finite
(3) the Newton polygon of the last invariant factor λ_d of the matrix D has positive slopes only.

Theorem 7.4. [M21, Theorem 7.2] *Let (E, Υ) be an abelian $\mathbf{t}$-module of dimension d and $D \in \mathrm{Mat}_{d\times d}(K\{\tau\})$ the matrix representing Υ_t with respect to a fixed coordinate system. Let M be the $\mathbf{t}$-motive of E. The $\mathbf{t}$-motive is pure if and only if the Newton polygon of the last invariant factor λ_d of D has exactly one edge.*

In this case, the weight of M equals the reciprocal of the slope of the edge.

Now we are ready to prove the following

Theorem 7.5. *Let $\Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big)$ be a triangular $\mathbf{t}$-module of dimension n. Then*

(i) Υ is finite and, therefore, also abelian,
(ii) Υ is pure if and only if $\mathrm{rk}\,\psi_1 = \cdots = \mathrm{rk}\,\psi_n$

Proof. Consider the matrix $C = tI_n - \Upsilon_t$ embedded in $\mathrm{Mat}_{n\times n}(K(\{\sigma\})[t])$. Assume it has the following form:

$$C = \begin{bmatrix} (t-\psi_{n,\sigma}) & 0 & \cdots & 0 & 0 \\ \xi_{n-1\times n} & (t-\psi_{n-1,\sigma}) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi_{2\times n} & \xi_{2\times n-1} & \cdots & (t-\psi_{2,\sigma}) & 0 \\ \xi_{1\times n} & \xi_{1\times n-1} & \cdots & \xi_{1\times 2} & (t-\psi_{1,\sigma}) \end{bmatrix} \tag{7.3}$$

where $\xi_{i\times j} = -\delta_{i\times j}$. Consider the first column of C. If $\xi_{k\times n} = 0$, for all $k = n-1, \dots, 1$ we proceed to the next column. Otherwise, let k_1 be the greatest k such that $\xi_{k\times n} \neq 0$. Interchange the rows $r_1 \leftrightarrow r_{n-k_1+1}$. Perform the following elementary operations on rows: $r_{n-k_1+1} - (t - \psi_{n,\sigma})\xi^{-1}_{k_1\times n} r_1$ and $r_j - \xi_{n-j+1\times n}\xi^{-1}_{k_1\times n} r_1$ for $j = n - k_1 + 1, \dots n$. This ensures that in the first column the only non-zero entry is $C_{1,1}$. Perform further the following elementary row operations $r_j + \xi_{n-1,j}\xi^{-1}_{n-1,k_1} r_i$ for $j = n - k_1 + 1, \dots, n$ and $i = 2, \dots n - k_1$. This ensures that all the entries $C_{i,j}$ below the diagonal for $j = 2, \dots k_1 - 1$ are constant polynomials in $K(\{\sigma\})[t]$. If $k_1 > 1$ perform the following column operations: $c_{k_1} - \xi_{n-j+1\times k_1}\xi^{-1}_{k_1,n} c_j$ for j=$k_1 + 1, \dots, n$. It is now clear that by using obvious elementary column operations we get $C_{1,i} = 0$ for $i > 1$, and then multiplying the first column by $C^{-1}_{1,1}$ we can ensure $C_{1,1} = 1$. Notice that notation is being slightly abused because after each elementary operation we denoted the resulting matrix again by C. After these reductions it has the following form:

$$C_{i,i} = \begin{cases} 1 & \text{if} \quad i = 1, \\ (t - \psi_{n,\sigma})\xi^{-1}_{k_1\times n}(t - \psi_{k_1,\sigma}) & \text{if} \quad i = n - k_1 + 1, \\ (t - \psi_{n-i+1,\sigma}) & \text{if} \quad i \neq 1 \text{ and } i \neq k_1 \end{cases} \tag{7.4}$$

and

$$C_{i,j} = \begin{cases} 0 & \text{for} \quad j > i \quad \text{or} \quad i = 1, \\ (\xi_1)_{k\times n-j+1} \in K(\{\sigma\}) & \text{for} \quad j = 2, \dots n-1, \quad k = j+1, \dots n. \end{cases}$$

Now, we repeat the same procedure as previously for the columns $j = 2, \dots, n-1$ to ensure that the resulting matrix C is in diagonal form. Thus $C = \operatorname{diag}\big[d_1(t), \dots, d_n(t)\big]$ where $d_i(t)$ is either 1 or is a product of linear factors and constants, i.e. $d_i(t) = (t - \psi_{s^i_1,\sigma})\beta_{i,1}(t - \psi_{s^i_2,\sigma})\beta_{i,2} \dots \beta_{i,k-1}(t - \psi_{s^i_{k_i},\sigma})$ where the disjoint union $\coprod_i \{s^i_1, \dots s^i_{k_i}\}$ is a partition of the set $\{1, \dots, n\}$. This is not a Smith normal form, since the divisibility property $d_1(t)|\dots|d_n(t)$ is not yet guaranteed. For this, we need the following lemma

Lemma 7.6. *Let $A \in \mathrm{Mat}_{2\times 2}(K(\{\sigma\})[t])$ be a matrix of the form*

$$A = \begin{bmatrix} p_1(t) & 0 \\ 0 & p_2(t) \end{bmatrix}$$

where $p_1(t)$ and $p_2(t)$ are non-zero polynomials. Then A can be reduced to the Smith normal form by means of at most five elementary operations.

Proof of Lemma 7.6. If $p_1(t)|p_2(t)$ then there is nothing to prove. The left division algorithm yields polynomials $a(t)$ and $b(t)$ such that the greatest (left) common divisor of $p_1(t)$ and $p_2(t)$ can be expressed as $d(t) = p_1(t)a(t) + p_2(t)b(t)$. We apply the following sequence of elementary operations: $c_2 + c_1 a(t)$, $r_1 + r_2 b(t)(t)$, $c_1 - c_2(p_1(t)/d(t) - 1)$, $c_2 - c_1$, $r_2 + r_1(p_2(t)/d(t))(p_1(t)/d(t) - 1)$, where since $K(\{\sigma\})$ is a skew field $p_1(t)/d(t)$

(resp. $p_2(t)/d(t)$) is a polynomial such that $d(t)(p_1(t)/d(t)) = p_1(t)$ (resp. $d(t)(p_2(t)/d(t)) = p_2(t)$). In this way, we obtain the following reduced matrix in Smith normal form:

$$A = \begin{bmatrix} d(t) & 0 \\ 0 & p_2(t)(p_1(t)/d(t)) \end{bmatrix} \tag{7.5}$$

□

Notice that by interchanging rows, and then corresponding columns, we can change the order of diagonal terms in $C = \mathrm{diag}[d_1(t), \dots, d_n(t)]$. So place all the constants (equal to 1) at the beginning of the sequence $d_1, \dots d_n$. Let k be a minimal index such that d_k is not a constant polynomial. Applying Lemma 7.6 to the blocs $\mathrm{diag}[d_k(t), d_j(t)]$, $j = k+1, \dots, n$, (embedded in C in an obvious way) we see that we can replace $d_k(t)$ by $\hat{d}_k(t) = \gcd(d_k(t), \dots, d_n(t))$ and $d_j(t)$ by $\hat{d}_j(t)$, $j = k+1, \dots, n$ such that $\hat{d}_k(t) | \hat{d}_j(t)$ for every $j = k+1, \dots, n$. Now, we repeat this process for $k+1, k+2, \dots, n-1$. As a result we obtain the Smith normal form of C i.e. $C = \mathrm{diag}[\lambda_1(t), \dots, \lambda_n(t)]$ with $\lambda_1 | \dots | \lambda_n$. Without loss of generality we may assume that $\lambda_n(t)$ is monic and therefore

$$\lambda_n(t) = \prod_{i=1}^{n} (t - \psi_i)$$

Now we will determine the Newton polygon of $\lambda_n(t)$. Notice that for a Drinfeld module ψ we have $\mathrm{ord}_\sigma(\psi) = -\mathrm{rk}\,\psi$. Notice that the coefficients $a_i(t)$ of $\lambda_n(t) = t^n + a_{n-1}(\sigma)t^{n-1} + a_1(\sigma)t + a_0(\sigma)$ are of the following form:

$$a_i = (-1)^{n-i} s_i(\psi_1, \dots, \psi_n) \quad i = 0, \dots, n-1, \tag{7.6}$$

where $s_i(x_1, \dots, x_n) = \sum_{1 \le k_1 < \dots < k_i \le n} x_1^{k_1} \cdot \dots \cdot x^{k_i}$ is the i-th elementary symmetric polynomial. We may assume that the ranks $r_i := \mathrm{rk}\,\psi_i$ fulfill the following inequalities: $r_1 \le r_2 \le \dots \le r_n$. Then the vertices of the Newton polygon of $\lambda_n(t)$ have the following coordinates: $P_i = \big(i, \mathrm{ord}_\sigma a_i\big) = \big(i, -\sum_{k=1}^{n-i} r_k\big)$ for $i = 0, \dots, n-1$ and $P_n = (n, 0)$.

Part (i) now follows easily from Theorem 7.3 and part (ii) from Theorem 7.4. □

Remark 7.1. Recently, A. Maurischat developed an algorithm [M25] for finding t-bases for both **t**-motives and **t**-comotives corresponding to finite **t**-modules. Theorem 7.5 (i) shows that his algorithm is applicable for triangular **t**-modules. Determining the t-bases of motives and comotives is necessary for effectively finding a rigid analytic trivialization once we know the period lattice of a **t**-module [NP24, section 4] (see also [HJ20, section 2]).

7.2. **Taelman's conjecture.** Let $K = \mathbb{F}_q(\theta)$ and let F be a finite extension of K. Let $\mathcal{O}_F$ be an integral closure of A in F and $F_\infty = F \otimes_K K_\infty$. Consider a $\mathbf{t}$-module Υ over $\mathcal{O}_F$, i.e. an $\mathbb{F}_q$-algebra homomorphism $\Upsilon : A \to \mathrm{Mat}_{d\times d}(\mathcal{O}_F)\{\tau\}$ such that for every $\Upsilon_a = \partial_\Upsilon(a) + \sum_{i=1}^n \Upsilon_{a,i}\tau^i$ one has $(\partial_\Upsilon(a) - aI_d)^d = 0$.

Set $W_\Upsilon(F_\infty) := \mathrm{Lie}(\Upsilon)(F_\infty)/(\partial_\Upsilon(\theta) - \theta I_d)\mathrm{Lie}(\Upsilon)(F_\infty)$ and let

$$w : \mathrm{Lie}(\Upsilon)(F_\infty) \to W_\Upsilon(F_\infty)$$

be the projection. In [An20] the authors consider the following form of Taelman's conjecture (cf. [Ta09]).

Conjecture 7.2. (Taelman's conjecture [An20]) With the above notation, suppose that Υ is finite and uniformizable. Then there exist an element $a \in A\backslash\{0\}$ and a sub-A-module $Z \subset \mathrm{Lie}(\Upsilon)(F_\infty)$ of rank $r := \dim_{K_\infty} W_\Upsilon(F_\infty)$ such that

1) $\exp_\Upsilon(Z) \subset \mathrm{Lie}(\Upsilon)(\mathcal{O}_F)$
2) $\bigwedge_A^r w(Z) = a \cdot L(\Upsilon/\mathcal{O}_F) \cdot \bigwedge_A^r W_\Upsilon(\mathcal{O}_F)$

Remark 7.3. In the above conjecture

$$L(\Upsilon/\mathcal{O}_F) := \prod_{\mathfrak{p}} \frac{[\mathrm{Lie}(\Upsilon)(\mathcal{O}_F/\mathfrak{p})]_A}{[\Upsilon(\mathcal{O}_F)/\mathfrak{p}]_A}$$

where $\mathfrak{p}$ runs through the set of maximal ideals of $\mathcal{O}_F$ and the right-hand side converges in K_∞ (cf. [Fa15]).

Since triangular $\mathbf{t}$-modules are finite and uniformizable, it is natural to ask whether the Taelman's conjecture holds for them. J. Fang [Fa15] proved that Conjecture 7.2 holds for $\mathbf{t}$-modules with no nilpotence. Thus, we have the following:

Corollary 7.7. *Let $\Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big)$ be a triangular $\mathbf{t}$-module defined over $\mathcal{O}_F$ with no nilpotence, i.e. none of $\delta_{i,j}$ appearing in $\underline{\delta}$ has a constant term. Then Conjecture 7.2, holds true.*

By [An20, Theorem C] we also have the following:

Corollary 7.8. *Let $\Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big)$ be a triangular $\mathbf{t}$-module defined over $\mathcal{O}_F$, and let C be the Carlitz module. Then Conjecture 7.2 holds for $\Upsilon \otimes C/\mathcal{O}_F$.*

8. Weil-Barsotti formula

In this section, we study the notion of duality for triangular t-modules. We introduce the concept of a dual biderivation and show its $\mathbb{F}_q$-linearity. We also establish an analogue of the classical Weil–Barsotti formula for triangular t-modules.

8.1. **Dual modules for triangular t-modules.** Let Φ and Ψ be $\mathbf{t}$-modules. Consider a biderivation $\delta \in \mathrm{Der}(\Phi, \Psi)$. We say that δ has no constant term if $\partial\delta = 0$. The space of all biderivations without constant term will be denoted as $\mathrm{Der}_0(\Phi, \Psi)$. By $\mathrm{Der}_{in,0}(\Phi, \Psi)$ we denote the space of inner biderivations with no constant term. An important example of an $\mathbb{F}_q[t]-$module is

$$\mathrm{Ext}_0(\Phi, \Psi) := \mathrm{Der}(\Phi, \Psi)/\mathrm{Der}_{in,0}(\Phi, \Psi) \tag{8.1}$$

for which there exists the following exact sequence of $F_q[t]-$modules

$$0 \longrightarrow \mathrm{Ext}_0(\Phi, \Psi) \longrightarrow \mathrm{Ext}^1_\tau(\Phi, \Psi) \longrightarrow \mathrm{Ext}^1\left(\mathrm{Lie}(E), \mathrm{Lie}(F)\right),$$

see [PR03, Corollary 2.3].

Let C be the Carlitz module, i.e. $C_t = \theta + \tau$. For a $\mathbf{t}$-module Φ, we define its dual module $\Phi^\vee$, in the following manner:

$$\Phi^\vee := \mathrm{Ext}_0(\Phi, C). \tag{8.2}$$

A dual module to a $\mathbf{t}$-module is an $\mathbb{F}_q[t]-$module, but it does not necessarily have a structure of a $\mathbf{t}$-module, see [KK26, Example 4.1]. We are interested in the cases where $\Phi^\vee$ is a $\mathbf{t}$-module. This is a case where Φ is a Drinfeld module of at least rank two [PR03, Theorem 1.1], or is a product of such Drinfeld modules [KK24, Corollary 6.2], or is a strictly pure $\mathbf{t}$-module of rank at least two [KK24, Theorem 8.4]. We will show that an analogous assertion for triangular $\mathbf{t}$-modules holds true .

We start with the following:

Lemma 8.1. *Let ψ be a Drinfeld module. Then $\psi^\vee$ is a $\tau-$simple module.*

Proof. Let $\psi = \theta + \sum_{i=1}^{r} a_i\tau^i$. Recall that $\psi^\vee$ has the following form:

$$\psi^\vee = \begin{bmatrix} \theta & 0 & 0 & \cdots & 0 & -\frac{a_1}{a_r}\tau + \frac{1}{a_r^{(1)}}\tau^2 \\ \tau & \theta & 0 & \cdots & 0 & -\frac{a_2}{a_r}\tau \\ 0 & \tau & \theta & \cdots & 0 & -\frac{a_3}{a_r}\tau \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \theta & -\frac{a_{r-2}}{a_r}\tau \\ 0 & 0 & 0 & \cdots & \tau & \theta - \frac{a_{r-1}}{a_r}\tau \end{bmatrix} \tag{8.3}$$

Notice that $\psi^\vee \cong \bigwedge^{r-1}\psi$. Assume that $\psi^\vee$ is not τ-simple. Thus there exists a nontrivial exact sequence of $\mathbf{t}$-motives:

$$0 \to M_1 \to \bigwedge^{r-1} M_\psi \to M_2 \to 0$$

where $\operatorname{rk} M_1, \operatorname{rk} M_2 < \operatorname{rk} \bigwedge^{r-1} M_\psi$ and $d(M_1), d(M_2) < d(\bigwedge^{r-1} M_\psi)$. But by [T95, Theorem 5.1] $w(\psi^\vee) = (r-1)/r$. Since $\gcd(r, r-1) = 1$ for $r = \operatorname{rk}\psi \geq 2$, (2.4) yields a contradiction. Thus $\psi^\vee$ is τ-simple. □

Notice that the concept of duality is consistent with the $\tau-$composition series.

Theorem 8.2. *Let $\Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big)$ be a triangular $\mathbf{t}$-module with no nilpotence such that $\operatorname{rk}\psi_i > 1$ for $i = 1, \ldots, n$.*
Then

(i) dual $\Upsilon^\vee$ is a $\mathbf{t}$-module given by the $\tau-$co-composition series of the following form:

$$\Upsilon_1^\vee \twoheadrightarrow \Upsilon_2^\vee \twoheadrightarrow \cdots \twoheadrightarrow \Upsilon_n^\vee = \Upsilon^\vee,$$

(ii) there exists the following exact sequence of $\mathbf{t}$-modules

$$0 \longrightarrow \Upsilon^\vee \longrightarrow \operatorname{Ext}^1_\tau(\Upsilon, C) \longrightarrow \mathbb{G}_a^n \longrightarrow 0$$

(iii) Every map $f : \Upsilon \longrightarrow \widehat{\Upsilon}$ of $\mathbf{t}$-modules that satisfy the assumptions of this theorem induces a map $f^\vee : \widehat{\Upsilon}^\vee \longrightarrow \Upsilon^\vee$.

Proof. Consider the following exact sequences:

$$\delta_l : \quad 0 \longrightarrow \Upsilon_l \longrightarrow \Upsilon_{l+1} \longrightarrow \psi_{l+1} \longrightarrow 0$$

for $l = 1, 2, \ldots, n-1$, where $\Upsilon_1 = \psi_1$ is a Drinfeld module. Notice that from Proposition 5.1 it follows that $\operatorname{Hom}_\tau(\Upsilon_l, C)$ vanishes for every $l = 1, 2, \ldots n-1$.

Applying the functor $\operatorname{Hom}_\tau(-, C)$ to the sequence δ_1 we obtain the following exact sequence of $\mathbb{F}_q[t]$-modules:

$$0 \longrightarrow \operatorname{Ext}^1_\tau(\psi_2, C) \longrightarrow \operatorname{Ext}^1_\tau(\Upsilon_2, C) \longrightarrow \operatorname{Ext}^1_\tau(\psi_1, C) \longrightarrow 0.$$

From [GKK24, Lamma 3.2], it follows that this sequence is also a sequence of $\mathbf{t}$-modules. The existence of the exact sequences of $\mathbf{t}$-modules (see [PR03, Theorem 1.1])

$$0 \longrightarrow \psi_l^\vee \longrightarrow \operatorname{Ext}^1_\tau(\psi_l, C) \longrightarrow \mathbb{G}_a \longrightarrow 0 \quad \text{for} \quad l = 1, 2$$

implies that there exists the following commutative diagram of $\mathbf{t}$-modules with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \psi_2^\vee \arrow[r] \arrow[d] & \Upsilon_2^\vee \arrow[r] \arrow[d] & \psi_1^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1_\tau(\psi_2, C) \arrow[r] \arrow[d] & \mathrm{Ext}^1_\tau(\Upsilon_2, C) \arrow[r] \arrow[d] & \mathrm{Ext}^1_\tau(\psi_1, C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{G}_a \arrow[r] \arrow[d] & \mathbb{G}_a \times \mathbb{G}_a \arrow[r] \arrow[d] & \mathbb{G}_a \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Therefore, we obtain the following exact sequence:

$$0 \longrightarrow \Upsilon_2^\vee \longrightarrow \mathrm{Ext}^1_\tau(\Upsilon_2, C) \longrightarrow \mathbb{G}_a^2 \longrightarrow 0.$$

Repeating this reasoning for the exact sequences δ_l for $l = 2, 3, \dots, n-1$ we obtain the fact that there are the following exact sequences of $\mathbf{t}$-modules:

$$\delta_l^\vee : \quad 0 \longrightarrow \psi_{l+1}^\vee \longrightarrow \Upsilon_{l+1}^\vee \longrightarrow \Upsilon_l^\vee \longrightarrow 0$$

for $l = 1, 2, \dots, n-1$. From these exact sequences, in the same way as before, we obtain the following exact sequences of $\mathbf{t}$-modules

$$0 \longrightarrow \Upsilon_l \longrightarrow \mathrm{Ext}^1_\tau(\Upsilon_l, C) \longrightarrow \mathbb{G}_a^l \longrightarrow 0$$

for $l = 2, \dots, n$. The last sequence for $l = n$ shows (ii).

From lemma 8.1 the modules $\psi^\vee$ are $\tau-$simple. Therefore, the sequences $\delta_l^\vee$ for $l = 1, 2, \dots, n-1$ show (i).

Part (iii). From (ii) there exist exact sequences of the following form:

$$\begin{aligned}
&0 \longrightarrow \Upsilon^\vee \longrightarrow \mathrm{Ext}^1_\tau(\Upsilon, C) \longrightarrow \mathbb{G}_a^n \longrightarrow 0 \\
&0 \longrightarrow \widehat{\Upsilon}^\vee \longrightarrow \mathrm{Ext}^1_\tau(\widehat{\Upsilon}, C) \longrightarrow \mathbb{G}_a^m \longrightarrow 0.
\end{aligned}$$

Arguing similarly as in the proof of [KK26, Proposition 3.2], we obtain that there exists the following commutative diagram with the exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Upsilon^\vee \arrow[r] \arrow[d, "f^\vee"] & \mathrm{Ext}^1_\tau(\Upsilon, C) \arrow[r] \arrow[d, "\overline{f}"] & \mathbb{G}_a^n \arrow[r] \arrow[d, "\partial f"] & 0 \, . \\
0 \arrow[r] & \widehat{\Upsilon}^\vee \arrow[r] & \mathrm{Ext}^1_\tau(\widehat{\Upsilon}, C) \arrow[r] & \mathbb{G}_a^m \arrow[r] & 0
\end{tikzcd}$$

□

Remark 8.1. If $\Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big)$ is a triangular $\mathbf{t}$-module fulfilling the assumptions of the theorem 8.2, then the dual module $\Upsilon_t^\vee$ has the following matrix form:

$$\Upsilon^\vee = \left(\begin{array}{cccccc} \boxed{\psi_n^\vee} & \boxed{\quad\delta_{n-1}^\vee\quad} & & & & \\ 0 & \boxed{\psi_{n-1}^\vee} & \boxed{\quad\delta_{n-2}^\vee\quad} & & & \\ \vdots & & \ddots & & \vdots & \\ 0 & 0 & \cdots & \boxed{\psi_3^\vee} & \boxed{\delta_2^\vee} & \\ 0 & 0 & \cdots & 0 & \boxed{\psi_2^\vee} & \boxed{\delta_1^\vee} \\ 0 & 0 & \cdots & 0 & 0 & \boxed{\psi_1^\vee} \end{array}\right), \tag{8.4}$$

In what follows, the following proposition will be useful:

Proposition 8.3.

(*i*) *If ψ is a Drinfeld module such that $\operatorname{rk}\psi > 1$ then*

$$\operatorname{Hom}_\tau\left(\psi^\vee, C\right) = 0.$$

(*ii*) *If $\Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big)$ is a triangular $\mathbf{t}$-module with no nilpotence such that $\operatorname{rk}\psi_i > 1$ for $i = 1, \ldots, n$, then*

$$\operatorname{Hom}_\tau(\Upsilon^\vee, C) = 0.$$

Proof. Let $u \in \operatorname{Hom}_\tau(\psi_{j+1}^\vee, C)$ be a nonzero morphism, i.e. $u\psi^\vee = Cu$. Let $\psi_{j+1}^\vee$ be of the form (8.3). Then $u = [u_1, u_2, \ldots, u_{r-1}] \in \operatorname{Mat}_{1\times r-1}\big(K\{\tau\}\big)$. The equality $u\psi^\vee = Cu$ yields the following relations:

$$u_i \cdot \theta + u_{i+1} \cdot \tau = (\theta + \tau) \cdot u_i \quad \text{for} \quad i = 1, 2, \ldots, r-2.$$

and

$$u_1 \cdot \Big(-\frac{a_1}{a_r}\tau + \frac{1}{a_r^{(1)}}\tau^2\Big) - u_2 \cdot \frac{a_2}{a_r}\tau - \cdots - u_{r-1} \cdot \frac{a_{r-1}}{a_r}\tau = (\theta + \tau) \cdot u_{r-1}.$$

Comparing the highest degree terms in the first $r-2$ equalities we obtain that

$$\deg_\tau u_1 = \deg_\tau u_2 = \cdots = \deg_\tau u_{r-1}.$$

But then by comparing the degrees in the last equality, we obtain a contradiction. Thus $\operatorname{Hom}_\tau(\psi_{j+1}^\vee, C) = 0$. This shows (*i*).

For the proof of (*ii*), recall that according to Theorem 8.2 there exist the following exact sequences:

$$\delta_l^\vee: \quad 0 \longrightarrow \psi_{l+1}^\vee \longrightarrow \Upsilon_{l+1}^\vee \longrightarrow \Upsilon_l^\vee \longrightarrow 0 \quad \text{for} \quad l = 1, 2, \ldots, n-1.$$

Acting on $\delta_1^\vee$ by the functor $\operatorname{Hom}_\tau(-, C)$, we obtain the six-term exact sequence [KK24, Theorem 10.2]:

$$0 \longrightarrow \operatorname{Hom}_\tau(\psi_1^\vee, C) \longrightarrow \operatorname{Hom}_\tau(\Upsilon_2^\vee, C) \longrightarrow \operatorname{Hom}_\tau(\psi_2^\vee, C) \longrightarrow \cdots$$

This implies $\operatorname{Hom}_\tau(\Upsilon_2^\vee, C) = 0$. Reasoning analogously for $\delta_l^\vee$, and $l = 2, 3, \ldots, n-2$ we obtain $\operatorname{Hom}_\tau(\Upsilon^\vee, C) = 0$.

□

8.2. **Dual biderivation.** Our next goal is to determine those triangular $\mathbf{t}$-modules for which the Cartier-Nishi theorem holds true, i.e. $(\Upsilon^\vee)^\vee \cong \Upsilon$. In order to do this, we have to determine the exact form of the $\mathbf{t}$-modules $\Upsilon^\vee$. This amounts to finding the biderivations $\delta_i^\vee$ for $i = 1, \ldots n-1$ in (8.4). First, we will investigate the map $\delta_i \longmapsto \delta_i^\vee$. However, our reasoning will be applied in much more general situation. We start with the following lemma.

Lemma 8.4. *Let* $\delta : 0 \longrightarrow \Psi \longrightarrow \Upsilon \longrightarrow \Phi \longrightarrow 0$ *be a short exact sequence given by a* δ *that splits. Then*

(*i*) *If* $\operatorname{Hom}_\tau(\Psi, \zeta) = 0$ *then the exact sequence of* $\mathbb{F}_q[t]-$*modules*

$$0 \longrightarrow \operatorname{Ext}^1_\tau(\Phi, \zeta) \longrightarrow \operatorname{Ext}^1_\tau(\Upsilon, \zeta) \longrightarrow \operatorname{Ext}^1_\tau(\Psi, \zeta) \longrightarrow 0 \tag{8.5}$$

also splits.

(*ii*) *If* $\operatorname{Hom}_\tau(\Psi, C) = 0$, *then the exact sequence of* $\mathbb{F}_q[t]-$*modules*

$$0 \longrightarrow \Phi^\vee \longrightarrow \Upsilon^\vee \longrightarrow \Psi^\vee \longrightarrow 0$$

splits as well.

Proof. We start with the proof of (i). Since the sequence δ splits, there exists a matrix of skew polynomials U such that $\delta = U\Psi - \Psi U$. Applying the functor $\operatorname{Hom}_\tau(-, \zeta)$ to the sequence δ by [KK24, Theorem 10.2] we obtain the short, exact sequence of the form (8.5). In the language of biderivations, it can be written as:

$$0 \longrightarrow \frac{\operatorname{Der}(\Phi, \zeta)}{\operatorname{Der}_{in}(\Phi, \zeta)} \xrightarrow{i} \frac{\operatorname{Der}(\Upsilon, \zeta)}{\operatorname{Der}_{in}(\Upsilon, \zeta)} \xrightarrow{\pi} \frac{\operatorname{Der}(\Psi, \zeta)}{\operatorname{Der}_{in}(\Psi, \zeta)} \longrightarrow 0,$$

where $i(\delta_1) = \big[-\delta_1, 0\big]$ and $\pi\Big(\big[\delta_1, \delta_2\big]\Big) = -\delta_2$. We will show that π is a retraction.

Let $g : \operatorname{Der}(\Psi, \zeta) \longrightarrow \operatorname{Der}(\Upsilon, \zeta)$ be the $\mathbb{F}_q$-linear map given by the following formula $g(\delta_2) = \big[\delta_2 U, -\delta_2\big]$. Additionally g maps inner biderivations into inner biderivations. Indeed, if $\delta_2 \in \operatorname{Der}_{in}(\Psi, \zeta)$, i.e. $\delta_2 = V\Psi - \zeta V$ for some matrix V, then $g(\delta_2) = \delta_{\Upsilon,\zeta}^{\big([VU, -V]\big)} \in \operatorname{Der}_{in}(\Upsilon, \zeta)$. So g induces homomorphism of $\mathbb{F}_q[t]$-modules $\overline{g} : \dfrac{\operatorname{Der}(\Psi, \zeta)}{\operatorname{Der}_{in}(\Psi, \zeta)} \longrightarrow \dfrac{\operatorname{Der}(\Upsilon, \zeta)}{\operatorname{Der}_{in}(\Upsilon, \zeta)}$. Moreover $\pi \circ \overline{g} = id$, so π is a retraction and the sequence (8.5) splits.

Since $g(\delta_2) \in \operatorname{Der}_0(\Upsilon, \zeta)$ for every $\delta_2 \in \operatorname{Der}_0(\Psi, \zeta)$, part (ii) follows from part (i).

□

Notice that a description of an exact sequence $0 \longrightarrow \Psi \xrightarrow{i} \Upsilon \xrightarrow{\pi} \Phi \longrightarrow 0$ depends on the choice of coordinate system in the module Υ. In [PR03] Papanikolas and Ramachandran fixed the choice of coordinates in such a way that $i = [0,1]^T$ is an embedding into the second coordinate and $\pi = [1,0]$ is a projection onto the first coordinate. In our work we followed this convention. However, one can assume that $i = [1,0]^T$is an embedding into the first coordinate and $\pi = [0,1]$ is a projection onto the second coordinate. [2] Then the $\mathbf{t}$–module Υ has the following bloc form:

$$\Upsilon = \left[\begin{array}{c|c} \Psi & \delta^{op} \\ \hline 0 & \Psi \end{array}\right]$$

In order to distinguish, this space of biderivations will be denoted as $\mathrm{Der}^{op}(\Phi,\Psi)$. Then the biderivations $\delta^{(U)} = U\Phi - \Psi U$, for some matrix U of skew polynomials correspond to splitting sequences. Analogously, the inner biderivations will be denoted as $\mathrm{Der}^{op}_{in}(\Phi,\Psi)$. Then there is an $F_q[t]-$module isomorphism

$$\mathrm{Ext}^1_\tau(\Phi,\Psi) \cong \frac{\mathrm{Der}^{op}(\Phi,\Psi)}{\mathrm{Der}^{op}_{in}(\Phi,\Psi)}$$

This notation will be useful further on.

Consider an exact sequence of $\mathbf{t}$-modules of the form $\delta: \quad 0 \longrightarrow \Psi \longrightarrow \Upsilon \longrightarrow \Phi \longrightarrow 0$ and a $\mathbf{t}$-module ζ. According to [GKK24, Lemma 3.5] if $\mathrm{Hom}_\tau(\Psi,\zeta) = 0$, $\mathrm{Ext}^1_\tau(\Psi,\zeta)$ and $\mathrm{Ext}^1_\tau(\Phi,\zeta)$ are $\mathbf{t}$-modules coming from the algorithm of $\mathbf{t}$-reduction then $\mathrm{Ext}^1_\tau(\Upsilon,\zeta)$ has the structure of a $\mathbf{t}$-module. Moreover, from the proof of [GKK24, Proposition 3.2] it follows that the $\mathbf{t}$-module structure of Υ is given by the following matrix:

$$\Pi_{\Upsilon,\zeta} = \left[\begin{array}{c|c} \Pi_{\Psi,\zeta} & \Pi_\delta \\ \hline 0 & \Pi_{\Phi,\zeta} \end{array}\right],$$

where $\Pi_\delta \in \mathrm{Der}^{op}\Big(\mathrm{Ext}^1_\tau(\Psi,\zeta), \mathrm{Ext}^1_\tau(\Phi,\zeta)\Big)$ and $\Pi_{*,\zeta}$ is the matrix defining the $\mathbf{t}$-module structure of $\mathrm{Ext}^1_\tau(*,\zeta)$.

Define the following maps:

(i) $\Pi_{(-)} : \mathrm{Der}(\Phi,\Psi) \longrightarrow \mathrm{Der}^{op}\Big(\mathrm{Ext}^1_\tau(\Psi,\zeta), \mathrm{Ext}^1_\tau(\Phi,\zeta)\Big)$ defined by the assignment $\delta \longmapsto \Pi_\delta$,

(i) $\Pi^{op}_{(-)} : \mathrm{Der}^{op}(\Phi,\Psi) \longrightarrow \mathrm{Der}\Big(\mathrm{Ext}^1_\tau(\Psi,\zeta), \mathrm{Ext}^1_\tau(\Phi,\zeta)\Big)$ defined by the assignment $\delta^{op} \longmapsto \Pi^{op}_{\delta^{op}}$.

Lemma 8.5. *Let Φ, Ψ and ζ be $\mathbf{t}$-modules such that $\mathrm{Hom}_\tau(\Psi,\zeta) = 0$ and both $\mathrm{Ext}^1_\tau(\Psi,\zeta)$ and $\mathrm{Ext}^1_\tau(\Phi,\zeta)$ are $\mathbf{t}$-modules coming from the algorithm of $\mathbf{t}$-reduction. Then the maps $\Pi_{(-)}$ i $\Pi^{op}_{(-)}$ are $\mathbb{F}_q$-linear.*

[2]Such a convention for description of extensions is common in the theory of representations of algebras, see [R76, R98, Mel07, KM20] and many others.

Proof. We will give a proof for the map $\Pi_{(-)}$. The proof for $\Pi^{op}_{(-)}$ is analogous. Let $\dim \Phi = d$, $\dim \Psi = f$ and $\dim \zeta = f$.

Let $\delta_1, \delta_2 \in \mathrm{Der}(\Phi, \Psi)$. First, we will show that $\Pi_{(\delta_1+\delta_2)} = \Pi_{\delta_1} + \Pi_{\delta_2}$. Denote by Υ_i for $i = 1, 2$ the middle term of the exact sequence corresponding to the biderivation δ_i and denote by Υ_+ the middle term of the exact sequence corresponding to $\delta_1 + \delta_2$. Since both $\mathbf{t}$-module structures for $\mathrm{Ext}^1_\tau(\Psi, \zeta)$ and $\mathrm{Ext}^1_\tau(\Phi, \zeta)$ come from the algorithm of $\mathbf{t}$-reduction there exist the following bases:

$$U^{\Phi,\zeta}_{i\times j} \in \mathrm{Mat}_{f\times d}\big(K\{\tau\}\big) \quad \text{and} \quad U^{\Psi,\zeta}_{i\times l} \in \mathrm{Mat}_{f\times e}\big(K\{\tau\}\big)$$

which fulfill **step 2.** of the $\mathbf{t}$-reduction algorithm. From the proof of [GKK24, Lemma 3.5], it follows that the basis

$$\Big[U^{\Phi,\zeta}_{i\times j} \mid 0\Big], \quad \Big[0 \mid U^{\Psi,\zeta}_{i\times j}\Big]$$

fulfills **step 2.** of the $\mathbf{t}$-reduction algorithm. Therefore, the spaces $\mathrm{Ext}^1_\tau(\Upsilon_*, \zeta)$ are $\mathbf{t}$-modules for $* = 1, 2, +$. Now, the argument is similar to that of [GKK24, Proposition 3.2]. Let $\delta^{(U)}_*$ denote the inner biderivation belonging to $\mathrm{Der}_{in}(\Upsilon_*, \zeta)$ for $* = 1, 2, +$.

$$\begin{aligned}
\delta_*^{\left(\mu\tau^k\left[U^{\Phi,\zeta}_{i\times j}|0\right]\right)} &= \Big[\delta_{\Phi,\zeta}^{\left(\mu\tau^k U^{\Phi,\zeta}_{i\times j}\right)} \mid 0\Big] \quad \text{for} \quad * = 1, 2, + \\
\delta_*^{\left(\mu\tau^k\left[0|U^{\Psi,\zeta}_{i\times l}\right]\right)} &= \Big[\mu\tau^k U^{\Psi,\zeta}_{i\times l} \cdot \delta_* \mid \delta_{\Psi,\zeta}^{\left(\mu\tau^k U^{\Psi,\zeta}_{i\times l}\right)}\Big] \quad \text{for} \quad * = 1, 2 \\
\delta_+^{\left(\mu\tau^k\left[0|U^{\Psi,\zeta}_{i\times l}\right]\right)} &= \Big[\mu\tau^k U^{\Psi,\zeta}_{i\times l} \cdot (\delta_1 + \delta_2) \mid \delta_{\Psi,\zeta}^{\left(\mu\tau^k U^{\Psi,\zeta}_{i\times l}\right)}\Big],
\end{aligned}$$

where $\delta^{(U)}_{\star,\zeta} \in \mathrm{Der}_{in}(\star, \zeta)$. Since we are interested only in the terms $\Pi_{\delta_1}, \Pi_{\delta_2}$ and $\Pi_{\delta_1+\delta_2}$ we compute the action

$$t * \big[0 \mid E_{i\times l} c\tau^k\big] = \zeta_t \cdot \big[0 \mid E_{i\times l c\tau^k}\big] = \big[0 \mid \zeta_t \cdot E_{i\times l c\tau^k}\big].$$

In order to determine the $\mathbf{t}$-module structure on $\mathrm{Ext}^1_\tau(\Upsilon_*, \zeta)$, reducing by means of inner biderivations, we have to ensure that

$$\big[0 \mid \zeta_t \cdot E_{i\times l} c\tau^k\big] \in \mathrm{Mat}_{f\times d}\big(K\{\tau\}\big)_{<N} \times \mathrm{Mat}_{f\times e}\big(K\{\tau\}\big)_{<M},$$

where $N = \Big[\deg_\tau \delta^{\left(\mu\tau^0 U^{\Phi,\zeta}_{i\times j}\right)}\Big]_{i\times j}$ and $M = \Big[\deg_\tau \delta^{\left(\mu\tau^0 U^{\Psi,\zeta}_{i\times l}\right)}\Big]_{i\times l}$. This means that the degrees of the terms in the first bloc at the $i \times j$-th position are smaller than $\deg_\tau \delta^{\left(\mu\tau^0 U^{\Phi,\zeta}_{i\times j}\right)}$, and in the second bloc at the $i \times l$-th position are smaller than $\deg_\tau \delta^{\left(\mu\tau^0 U^{\Psi,\zeta}_{i\times l}\right)}$. Since the basis $U^{\Psi,\zeta}_{i\times l}$ fulfills **step 2.** of the $\mathbf{t}$-reduction algorithm, there exists a sequence of inner biderivations of the form

$$\delta_{\Psi,\zeta}^{\left(\mu_s\tau^{k_s} U^{\Psi,\zeta}_{i_s\times l_s}\right)} \quad \text{for} \quad s \in J$$

such that

$$\zeta_t \cdot E_{i\times l} c\tau^k - \sum_{s\in J} \delta_{\Psi,\zeta}^{\left(\mu_s\tau^{k_s}U^{\Psi,\zeta}_{i_s\times l_s}\right)} \in \mathrm{Mat}_{f\times e}\big(K\{\tau\}\big)_{<M}.$$

Therefore, in all three cases we will reduce by means of a sequence of biderivations:

$$\delta_*^{\left(\mu_s\tau^{k_s}\left[0|U^{\Psi,\zeta}_{i_s\times l_s}\right]\right)} \quad \text{for} \quad s\in J \quad \text{and} \quad *=1,2,+.$$

For Π_{δ_i}, $i=1,2$, we obtain

$$t * \left[0 \mid E_{i\times l}c\tau^k\right] = \left[-\sum_{s\in J}\mu_s\tau^{k_s}U_{i_s\times l_s}\cdot \delta_i\right| \quad \text{reduced part}\Big].$$

For $i=1,2$ denote $w_i(\tau) := -\sum_{s\in J}\mu_s\tau^{k_s}U^{\Psi,\zeta}_{i_s\times l_s}\cdot\delta_i$. Since the basis $U^{\Phi,\zeta}_{i\times j}$ fulfills **step 2.** of the **t**-reduction algorithm, there exists a sequence of inner biderivations of the form

$$\delta_{\Phi,\zeta}^{\left(\mu_r\tau^{k_r}U^{\Phi,\zeta}_{i_r\times j_r}\right)} \quad r\in I_i \quad \text{for} \quad i=1,2$$

such that

$$w_i(\tau) - \sum_{r\in I_i}\delta_{\Phi,\zeta}^{\left(\mu_r\tau^{k_r}U^{\Phi,\zeta}_{i_r\times j_r}\right)} \in \mathrm{Mat}_{f\times d}\big(K\{\tau\}\big)_{<N}.$$

Therefore, in the reduction process, we use inner biderivations of the following form:

$$\delta_i^{\left(\mu_r\tau^{k_r}\left[U^{\Phi,\zeta}_{i_r\times j_r}|0\right]\right)} \quad r\in I_i \quad \text{for} \quad i=1,2.$$

In the case of $\Pi_{\delta_1+\delta_2}$ we have

$$\begin{aligned} t * \left[0 \mid E_{i\times l}c\tau^k\right] &= \left[-\sum_{s\in J}\mu_s\tau^{k_s}U_{i_s\times l_s}\cdot(\delta_1+\delta_2)\right| \quad \text{reduced part}\Big] \\ &= \Big[w_1(\tau)+w_2(\tau)\Big| \quad \text{reduced part}\Big]. \end{aligned}$$

Then for the reduction we can use the former sequences

$$\delta_1^{\left(\mu_r\tau^{k_r}\left[U^{\Phi,\zeta}_{i_r\times j_r}|0\right]\right)} \quad r\in I_1 \quad \text{and} \quad \delta_2^{\left(\mu_r\tau^{k_r}\left[U^{\Phi,\zeta}_{i_r\times j_r}|0\right]\right)} \quad r\in I_2.$$

Since for $\Pi_{\delta_1+\delta_2}$ we used the same inner biderivations as for Π_{δ_1} and Π_{δ_2}, we obtain the same reduced form. This shows that $\Pi_{\delta_1+\delta_2} = \Pi_{\delta_1}+\Pi_{\delta_2}$.

In order to show that $x\cdot\Pi_\delta = \Pi_{x\cdot\delta}$ for $x\in\mathbb{F}_q$, we can proceed similarly. First, we determine the inner biderivations for Π_δ and $\Pi_{x\cdot\delta}$. Then we show that for computation of the action $t * \left[0 \mid E_{i\times l}c\tau^k\right]$, we can use the same inner biderivations. We leave the details to the reader.

□

From Lemmas 8.4 and 8.5 we obtain the following theorem.

Theorem 8.6. *Let Φ, Ψ and ζ be* $\mathbf{t}$*-modules such that* $\mathrm{Hom}_\tau(\Psi, \zeta) = 0$ *and both* $\mathrm{Ext}^1_\tau(\Psi, \zeta)$ *and* $\mathrm{Ext}^1_\tau(\Phi, \zeta)$ *are* $\mathbf{t}$*-modules coming from the* $\mathbf{t}$*-reduction algorithm.*

(i) Then the following maps:

$$\Pi_{(-)}, \Pi^{op}_{(-)} : \mathrm{Ext}^1_\tau(\Phi, \Psi) \longrightarrow \mathrm{Ext}^1_\tau \Big(\mathrm{Ext}^1_\tau(\Psi, \zeta), \mathrm{Ext}^1_\tau(\Phi, \zeta)\Big)$$

are $\mathbb{F}_q$*-linear.*

(ii) If the $\mathbf{t}$*-modules* Φ, Ψ *and* ζ *have no nilpotency then the above maps induce the following* $\mathbb{F}_q-$*linear maps:*

$$\Pi_{(-)}, \Pi^{op}_{(-)} : \mathrm{Ext}_0(\Phi, \Psi) \longrightarrow \mathrm{Ext}_0 \Big(\mathrm{Ext}_0(\Psi, \zeta), \mathrm{Ext}_0(\Phi, \zeta)\Big)$$

Proof. From additivity of $\Pi_{(-)}$ and $\Pi^{op}_{(-)}$ and the fact that they map inner biderivations into inner biderivations, it follows that they induce the maps $\mathrm{Ext}^1_\tau(\Phi, \Psi) \to \mathrm{Ext}^1_\tau \Big(\mathrm{Ext}^1_\tau(\Psi, \zeta), \mathrm{Ext}^1_\tau(\Phi, \zeta)\Big)$. Linearity of these maps is the assertion of Lemma 8.5 which shows (i). We will prove (ii) for the map $\Pi_{(-)}$. The argument for $\Pi^{op}_{(-)}$ is analogous. It is enough to show that if $\delta \in \mathrm{Der}(\Phi, \Psi)$ fulfills $\partial\delta = 0$, then also $\partial\Pi_{(\delta)} = 0$. But this follows directly from the $\mathbf{t}$-reduction algorithm, since if both Υ and ζ are $\mathbf{t}$-modules with no nilpotence, then the $\mathbf{t}$-module $\mathrm{Ext}^1_\tau(\Upsilon, \zeta)$ also has no nilpotence. □

Corollary 8.7. *Let* Φ *and* Ψ *be* $\mathbf{t}$*-modules with no nilpotence such that* $\mathrm{Hom}_\tau(\Psi, C) = 0$ *and both* $\mathrm{Ext}^1_\tau(\Psi, C)$ *and* $\mathrm{Ext}^1_\tau(\Phi, C)$ *are* $\mathbf{t}$*-modules coming from the* $\mathbf{t}$*-reduction algorithm. Then the maps*

$$\Pi_{(-)}, \Pi^{op}_{(-)} : \mathrm{Ext}_0(\Phi, \Psi) \longrightarrow \mathrm{Ext}_0 \Big(\Psi^\vee, \Phi^\vee \Big)$$

are $\mathbb{F}_q-$*linear.*

For Corollary 8.7 we will write $\delta^\vee$ instead of Π_δ and call it a biderivation dual to δ. If $\Psi^\vee$ and $\Phi^\vee$ again fulfill the assumptions of Corollary 8.7 then we can consider the following composition of maps

$$\mathrm{Ext}_0(\Phi, \Psi) \xrightarrow{\Pi_{(-)}} \mathrm{Ext}_0 \Big(\Psi^\vee, \Phi^\vee \Big) \xrightarrow{\Pi^{op}_{(-)}} \mathrm{Ext}_0 \Big(\big(\Phi^\vee\big)^\vee, \big(\Psi^\vee\big)^\vee \Big),$$

which will be denoted as

$$\delta \longmapsto \big(\delta^\vee\big)^\vee.$$

For finding an exact formula for a dual module to a triangular $\mathbf{t}$-module we need the following corollary. Applying this corollary considerably simplifies the procedure of finding the aforementioned dual module.

.

Corollary 8.8. *Let* $\Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big)$ *be a triangular* $\mathbf{t}$*-module with no nilpotence of the form* (4.3) *where* $\mathrm{rk}\,\psi_i > 1$*. Then* *(i) the correspondence* $\delta_i \longmapsto \delta_i^\vee$ *is* $\mathbb{F}_q-$*linear.*

(ii) the correspondence $\delta_i \longmapsto \left(\delta_i^\vee\right)^\vee$ *is* $\mathbb{F}_q-$*linear.*

Proof. Assume that Υ has a $\tau-$composition series of the form (4.1). Then every pair of **t**-modules ψ_{j+1} and Υ_j for $j = 1, \dots n-1$ meets the assumptions of Corollary 8.7. From Proposition 5.1 we obtain that $\mathrm{Hom}_\tau(\Upsilon, C)$ vanishes since $\mathrm{rk}\, \psi_i > 1$ for all $i = 1, 2, \dots, n$. Moreover, by [GKK24, Corollary 5.3.] we see that both $\mathrm{Ext}^1_\tau(\psi_{j+1}, C)$ and $\mathrm{Ext}^1_\tau(\Upsilon_j, C)$ are **t**-modules coming from the **t**-reduction algorithm. Then part (i) follows by induction applied to Corollary 8.7.

By Theorem 8.2 the **t**-module $\Upsilon^\vee$ has a $\tau-$ cocomposition series of the form (4.2). Now we will show that the pairs $\Upsilon_j^\vee$, $\psi_{j+1}^\vee$ fulfill the assumptions of Corollary 8.7.

By Proposition 8.3 the space $\mathrm{Hom}_\tau(\psi_{j+1}^\vee, C)$ vanishes.

So the fact that $\mathrm{Ext}^1_\tau(\psi_{j+1}^\vee, C)$ and $\mathrm{Ext}^1_\tau(\Upsilon_j^\vee, C)$ are **t**-modules coming from the **t**-reduction algorithm follows from Lemma 9.2 which will be proven in the next section. The proof of (ii) again then follows by induction applied to Corollary 8.7. □

Remark 8.2. Notice an important fact that $\mathbb{F}_q-$linearity is valid only for a specific biderivation δ_i, not for different biderivations. This means that for a determination of a dual module to that of the following form:

$$\Upsilon = \begin{bmatrix} \theta+\tau^3 & 0 & 0 \\ \tau^4+\tau^5 & \theta+\tau^4 & 0 \\ \tau+\tau^2 & \tau^5+\tau^6 & \theta+\tau^2 \end{bmatrix},$$

we have to proceed as follows. First, we compute a dual module to the following **t**-module:

$$\begin{bmatrix} \theta+\tau^4 & 0 \\ \tau^5+\tau^6 & \theta+\tau^2 \end{bmatrix}$$

by applying additivity $\left(\tau^5+\tau^6\right)^\vee = \left(\tau^5\right)^\vee + \left(\tau^6\right)^\vee$. Then we compute the dual module to Υ by determining the dual modules for the following **t**-modules:

$$\begin{bmatrix} \theta+\tau^3 & 0 & 0 \\ \tau^4 & \theta+\tau^4 & 0 \\ \tau & \tau^5+\tau^6 & \theta+\tau^2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \theta+\tau^3 & 0 & 0 \\ \tau^5 & \theta+\tau^4 & 0 \\ \tau^2 & \tau^5+\tau^6 & \theta+\tau^2 \end{bmatrix}$$

where we use the following $\mathbb{F}_q-$linearity:

$$\begin{pmatrix} \tau^4+\tau^5 \\ \tau+\tau^2 \end{pmatrix}^\vee = \begin{pmatrix} \tau^4 \\ \tau \end{pmatrix}^\vee + \begin{pmatrix} \tau^5 \\ \tau^2 \end{pmatrix}^\vee$$

8.3. **Explicit formulas for dual modules and dual biderivations.** In the next section, we address the problem relating to which triangular $\mathbf{t}$-modules the Cartier-Nishi theorem holds true for, i.e. whether a double dual of a triangular $\mathbf{t}$-module is isomorphic to an initial $\mathbf{t}$-module. It turns out that for $\mathbf{t}$-modules with nilpotence the answer is negative.

We establish the Cartier-Nishi Theorem for $\mathbf{t}$-modules allowing for LD-biderivations. In order to prove this theorem, we need to find exact forms of dual biderivations for this case. This will again be achieved by applying the $\mathbf{t}$-reduction algorithm from [GKK24]. We start with the following lemma.

Lemma 8.9. *Let $\Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big)$ be a triangular $\mathbf{t}$-module of dimension n with no nilpotence and $r_j := \operatorname{rk}\psi_j > 1$ for $j = 1, 2, \ldots, n$. Then there is an F_q-linear isomophism*

$$\Upsilon^{\vee} \cong \Big[K\{\tau\}_{\langle 1, r_n)}, \cdots, K\{\tau\}_{\langle 1, r_1)}\Big], \tag{8.6}$$

where $K\{\tau\}_{\langle 1, r_i)}$ denotes the space of polynomials with monomials of degrees belonging to the interval $\langle 1, r_i)$.

Proof. Recall that $\Upsilon^{\vee} = \operatorname{Ext}_0(\Upsilon, C)$ is isomorphic to $\operatorname{Der}_0(\Upsilon, C)/\operatorname{Der}_{in,0}(\Upsilon, C)$ as an $\mathbb{F}_q[t]-$module. Because of an isomorphismm $\operatorname{Der}(\Upsilon, C) \cong \operatorname{Mat}_{1\times n}(K\{\tau\})$ every biderivation $\delta \in \operatorname{Der}(\Upsilon, C)$ can be regarded as a $1\times n$ matrix of twisted polynomials.

In what follows, to simplify the notation, we will enumerate columns from right to left. According to this convention, E_i will denote the matrix of dimensions $1 \times n$, where entry at position $1 \times n - i + 1$ equals 1 and all other entries are 0.

For the proof, we use the following inner biderivations from $\operatorname{Der}_{in}(\Upsilon, C)$:

$$\begin{aligned}
\delta_{\Upsilon,C}^{(\mu\tau^k E_n)} &= \Big[\delta_{\psi_n,C}^{(\mu\tau^k)} \;\Big|\; 0 \;\Big|\; \cdots \;\Big|\; 0\Big] \\
\delta_{\Upsilon,C}^{(\mu\tau^k E_{n-1})} &= \left[\; \sum_{l=1}^{\deg\delta_{n-1}} \mu\cdot d_{n-1\times n,l}^{(k)}\tau^{k+l} \;\middle|\; \delta_{\psi_{n-1},C}^{(\mu\tau^k)} \;\middle|\; 0 \;\middle|\; \cdots \;\middle|\; 0\right] \\
\delta_{\Upsilon,C}^{(\mu\tau^k E_{n-2})} &= \left[\; \sum_{l=1}^{\deg\delta_{n-1}} \mu\cdot d_{n-2\times n,l}^{(k)}\tau^{k+l} \;\middle|\; \sum_{l=1}^{\deg\delta_{n-2}} \mu\cdot d_{n-2\times n-1,l}^{(k)}\tau^{k+l} \;\middle|\right. \\
&\qquad\qquad\qquad \left.\middle|\; \delta_{\psi_{n-2},C}^{(\mu\tau^k)} \;\middle|\; 0 \;\middle|\; \cdots \;\middle|\; 0\right] \\
&\;\;\vdots
\end{aligned} \tag{8.7}$$

$$\delta^{(\mu\tau^k E_1)}_{\Upsilon,C} = \Bigg[\sum_{l=1}^{\deg \delta_{n-1}} \mu \cdot d^{(k)}_{1\times n,l}\tau^{k+l} \;\Bigg|\; \sum_{l=1}^{\deg \delta_{n-2}} \mu \cdot d^{(k)}_{1\times n-1,l}\tau^{k+l} \;\Bigg|\; \cdots$$
$$\cdots \Bigg| \sum_{l=1}^{\deg \delta_1} \mu \cdot d^{(k)}_{1\times 2,l}\tau^{k+l} \;\Bigg|\; \delta^{(\mu\tau^k)}_{\psi_1,C} \Bigg],$$

where $\delta^{(\mu\tau^k)}_{\psi_l,C} = \mu\tau^k \cdot \psi_l - C \cdot \mu\tau^k \in \mathrm{Der}_{in}(\psi_l, C)$ for $l = 1, 2, \ldots, n$. Since Υ has no nilpotence, then all these biderivations belong to $\mathrm{Der}_0(\Upsilon, C)$. Notice that the inner biderivation $\delta^{(\mu\tau^k E_s)}$ will be used for reduction of terms at the $s-$th position counted from right to left. Similarly, every biderivation $\delta \in \mathrm{Der}_0(\Upsilon, C)$ will be reduced from the right to the left. Therefore, starting with an arbitrary biderivation $\delta \in \mathrm{Der}_0(\Upsilon, C)$ one can, by means of the inner biderivations $\delta^{(\mu\tau^k E_l)}$, reduce δ in such a way that the polynomial at the s-th place (counted from right to left) in the biderivation δ after reduction has degree less than $\mathrm{rk}\,\psi_s$. □

From Lemma 8.9 it follows that the basis E_n,..., E_2, E_1 satisfies **step 2.** of the **t**-reduction algorithm. Thus $\mathrm{Ext}^1_\tau\big(\Upsilon^\vee, C\big)$ has a **t**-module structure coming from the **t**-reduction algorithm. We also have the following proposition.

Proposition 8.10. *Let $\Upsilon = \big(\underline{\psi}, \underline{\delta}\big)$ be a triangular* **t***-module with no nilpotence, such that $\mathrm{rk}\,\psi_l > 1$ for $l = 1, 2, \ldots, n$. Then* **t***-module $\Upsilon^\vee$ has no nilpotence.*

Proof. Notice that if Υ has no nilpotence then the inner biderivations (8.7) satisfy the conditions $\partial\delta^{(\mu\tau^k E_j)} = 0$ for $j = 1, 2, \ldots n$. Therefore the **t**-module $\mathrm{Ext}^1_\tau\big(\Upsilon^\vee, C\big)$ has no nilpotence as well as the **t**-module $\Upsilon^\vee \subset \mathrm{Ext}^1_\tau\big(\Upsilon^\vee, C\big)$. □

Remark 8.3. In the process of determining a dual module as well as a double dual module, we will use Corollary 8.8 as in Remark 8.2. So we need to ensure that if a **t**-module Υ allows for LD-biderivations (c.f. Definition 4.4), then auxiliary **t**-modules coming from monomials of the biderivations for Υ also have this property. And indeed this is true. The converse is also true once we limit ourselves to the summation of monomials in one column, i.e. those within one biderivation.

In the proposition below we determine an explicit formula for the dual module in the case of a **t**-module allowing for LD-biderivations.

Proposition 8.11. *Let $\Upsilon = \Upsilon\big(\underline{\psi}, \underline{\delta}\big)$ be a triangular* **t***-module of dimension n, with no nilpotence and $r_j := \mathrm{rk}\,\psi_j > 1$ for $j = 1, 2, \ldots, n$. If Υ allows for*

LD-biderivations, then $\delta_i^\vee$ has the following bloc matrix form over $K(\Upsilon)$

$$\delta_i^\vee = \Big[D_{i+1,i} \mid \cdots \mid D_{i+1,2} \mid D_{i+1,1} \Big], \tag{8.8}$$

where $D_{i+1,j}$ is a matrix of size $r_i - 1 \times r_j - 1$ for $j = 1, 2 \ldots, i$.
Moreover, if the biderivations δ_i are of the form (4.4), then

$$D_{i+1,j} = -\frac{1}{a_{j,r_j}} \begin{bmatrix} 0 & \cdots & 0 & d_{j\times i+1,1} \\ 0 & \cdots & 0 & d_{j\times i+1,2} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & d_{j\times i+1,\deg \delta_{i+1}} \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} \tau, \tag{8.9}$$

where a_{j,r_j} is the leading term of ψ_j.

Proof. Since Υ allows for LD-biderivations, after changing to the field $K(\Upsilon)$, we may assume that Υ satisfies condition (4.6).

Determining the dual biderivations $\delta_i^\vee$ for $i = 1, 2, \cdots n-1$ will be done inductively. In the proof we will use the inner biderivations of the form (8.7).

The starting point of induction is determination of $\delta_1^\vee$. Let $\Upsilon_2 = \left[\begin{array}{c|c} \psi_2 & 0 \\ \hline \delta_1 & \psi_1 \end{array} \right]$, where $\operatorname{rk} \psi_i = r_i > 1$ for $i = 1, 2$. From Corollary 8.8 it follows that the correspondence $\delta_1 \longmapsto \delta_1^\vee$ is $\mathbb{F}_q-$linear. Therefore it is enough to find the dual biderivation for generators of the form $d\tau^h$ for $1 \le h < r_2$. Lemma 8.9 yields an isomorphism of $\mathbb{F}_q-$linear spaces:

$$\Upsilon^\vee \cong V_2 = \Big[K\{\tau\}_{\langle 1, r_2 \rangle}, K\{\tau\}_{\langle 1, r_1 \rangle} \Big].$$

Recall that, following the convention from Lemma 8.9, we denote $E_2 = [1, 0]$ and $E_1 = [0, 1]$ since we are counting terms from left to right.

After choosing the following coordinate system for V_2:

$$\Big(E_2 \tau^k \Big)_{k=1}^{r_2 - 1}, \Big(E_1 \tau^k \Big)_{k=1}^{r_1 - 1},$$

we can determine the $\mathbf{t}$-module structure on $\Upsilon^\vee$. To do this, one has to compute the results of the action of t on the generators $c\tau^k E_i$ for $i = 2, 1$, and express the obtained results in the chosen coordinate system.

Since we are only interested in determination of $\delta_i^\vee$ we will compute the action of t solely on these generators, which have influence on the form of $\delta_i^\vee$.

If $k \in \{0, 1, \ldots, r_1 - 2\}$, then

$$t * c\tau^k E_1 = (\theta + \tau) c\tau^k E_1 = \Big(\theta c \tau^k + c^{(1)} \tau^{k+1} \Big) E_1 \in V_2,$$

then $t * c\tau^k E_1$ in the fixed basis has the following coordinates:

$$\left[\ \mathbf{0}\ \Big|\ \cdots\ \Big|\ \mathbf{0}\ \Big|\underbrace{0,\dots,0,\theta,\tau,0,\dots,0}_{j-th\text{ block from right}}\Big|\ \cdots\ \Big|\ \mathbf{0}\ \right],$$

where $\mathbf{0}$ denotes the zero row of appropriate size. This shows that all terms except those in the last column of the matrix $\delta_1^\vee$ are zero.

Now consider the last column of $\delta_1^\vee$. The terms there result from the action of t on the generator $c\tau^{r_1-1}E_1$. Then

$$t * c\tau^{r_1-1}E_1 = (\theta+\tau)c\tau^{r_1-1}E_1 = \left(\theta c\tau^{r_1-1} + c^{(1)}\tau^{r_1}\right)E_1 \notin V_2.$$

Since the degree of $\theta c\tau^{r_1-1} + c^{(1)}\tau^{r_1}$ equals r_1 we reduce it by means of the following inner biderivation

$$\delta^{(\mu\tau^0 E_j)}_{\Upsilon,C} = \left[\mu d\tau^h \,\middle|\, \delta^{(\mu\tau^0)}_{\psi_1,C}\right] \quad \text{for} \quad \mu = \frac{c^{(1)}}{a_{1,r_1}}.$$

As a result one gets the following equality:

$$t * c\tau^{r_1-1}E_1 = \left[\ -\frac{c^{(1)}}{a_{1,r_1}}\cdot d\tau^h \ \middle|\ \text{terms for } \psi_1^\vee\ \right]$$

Since $h < r_1$ the value $t*c\tau^{r_1-1}E_1$ can be expressed in the aforementioned fixed basis. We obtain

$$t * c\tau^{r_1-1}E_1 = \left[\ 0,\dots,0,\underbrace{-\frac{d}{a_{1,r_1}}}_{h-th\text{ place}}\tau,0,\dots,0\ \middle|\ \text{terms for } \psi_1^\vee\ \right]\Bigg|_{\tau=c}.$$

Thus

$$\left(d\tau^h\right)^\vee = -\frac{d}{a_{1,r_1}}E_{h\times r_1-1}\tau,$$

where the matrix $E_{a\times b}$ has the only nonzero entry equal to one at the position $a\times b$.

Then for $\delta_1 = \sum_{k=0}^{\deg\delta_1} d_{1\times 2,k}\tau^k$ we obtain

$$\delta_1^\vee = \Big(\sum_{k=0}^{\deg\delta_1} d_{1\times 2,k}\tau^k\Big)^\vee = \sum_{k=0}^{\deg\delta_1}\left(d_{1\times 2,k}\tau^k\right)^\vee = \sum_{k=0}^{\deg\delta_1} -\frac{d_{1\times 2,k}}{a_{1,r_1}}E_{k\times r_1-1}\tau.$$

This finishes the determination of $\delta_1^\vee$ and the first step of induction.

Assume that the consecutive dual biderivations $\delta_l^\vee$ for $\mathbf{t}$-modules Υ_{l+1} where $l = 1,2,\dots,n-2$ have been determined and that they satisfy the following conditions:

- **a.** every $\delta_l^\vee$ is of the form (8.9),
- **b.** in the process of determining $\delta_l^\vee$, all the values $t * c\tau^k E_j$ gave the columns with entries equal to zero for $1 \le k < r_j - 1$ and $j = 1,2,\dots l-1$,

c. in the process of determining $\delta_l^\vee$ the values $t * c\tau^{r_j-1}E_j$ were obtained after a single reduction by the inner biderivation $\delta_{\Upsilon_l,C}^{(\mu\tau^0 E_j)}$ for $\mu = \dfrac{c^{(1)}}{a_{j,r_j}}$ and $j = 1, 2, \dots, l-1$.

Now we will find the form of the dual biderivation $\delta_{n-1}^\vee$ for the $\mathbf{t}$-module $\Upsilon = \Upsilon_n$. By the linearity of the correspondence $\delta_{n-1} \longmapsto \delta_{n-1}^\vee$, it is enough to find the dual biderivations for the generators of the following form:

$$\delta_{n-1} = \Big[d_{n-1}\tau^{h_{n-1}}, \dots, d_2\tau^{h_2}, d_1\tau^{h_1}\Big]^T.$$

Denote by V_n the space $\Big[K\{\tau\}_{\langle 1,\operatorname{rk}\psi_n\rangle}, \cdots, K\{\tau\}_{\langle 1,\operatorname{rk}\psi_1\rangle}\Big]$. By (8.6) we have $\Upsilon^\vee \cong V_n$. Then we have the following basis for V_n:

$$\Big(E_n\tau^k\Big)_{k=1}^{\operatorname{rk}\psi_n-1}, \cdots, \Big(E_2\tau^k\Big)_{k=1}^{\operatorname{rk}\psi_2-1}, \Big(E_1\tau^k\Big)_{k=1}^{\operatorname{rk}\psi_1-1} \tag{8.10}$$

Now we will find the form of the matrix $D_{n,j}$ for fixed $j \in \{1, 2, \dots, n-1\}$. Again, $t * c\tau^k E_j$ for $1 \le k < r_j - 1$ gives us zero-value columns.

Consider now the last column of $D_{n,j}$. The terms there come from the action of t on the generator $c\tau^k E_j$ for $k = r_j - 1$. Then

$$t * c\tau^{r_j-1}E_j = (\theta + \tau)c\tau^{r_j-1}E_j = \Big(\theta c\tau^{r_j-1} + c^{(1)}\tau^{r_j}\Big)E_j \notin V_n.$$

Since the degree of $\theta c\tau^{r_j-1} + c^{(1)}\tau^{r_j}$ equals $\operatorname{rk}\psi_j$ we reduce it by means of the following inner biderivation:

$$\delta_{\Upsilon,C}^{(\mu\tau^0 E_j)} = \Big[\ \mu d_j\tau^{h_j}\ \Big|\ \delta_{\Upsilon_{n-1},C}^{(\mu\tau^0 E_j)}\ \Big] \quad \text{for} \quad \mu = \frac{c^{(1)}}{a_{j,r_j}}.$$[3]

As a result, one gets the following equality:

$$t * c\tau^{r_j-1}E_j = \Big[\ -\mu d_j\tau^{h_j}\ \Big|\ \text{terms for } \Upsilon_{n-1}^\vee\ \Big] \in V_n, \tag{8.11}$$

where the right-hand side has, by inductive hypotheses, the summands for $\Upsilon_{n-1}^\vee$. Since $h_j < \operatorname{rk}\psi_j$, the value $t * c\tau^{r_j-1}E_j$ can be expressed in the basis (8.10). We obtain

$$t * c\tau^{r_j-1}E_j = \left[\ 0, \dots, 0, \underbrace{-\frac{d_j}{a_{j,r_j}}}_{h_j-th\ \text{place}}\tau, 0, \dots, 0\ \Big|\ \text{terms for } \Upsilon^\vee\ \right]\Bigg|_{\tau=c}.$$

Then the bloc $D_{n,j} = -\dfrac{d_j}{a_{j,r_j}}E_{h_j\times r_j-1}\tau$, which shows (8.9). □

[3]Notice that in the above formula we used the symbol E_j for two different matrices. In the left-hand side the matrix E_j is of type $1 \times n$ whereas in the right-hand side it is of type $1 \times (n-1)$. In both cases the only nonzero term is 1 at the j-th position counted from the left. This should not lead to any ambiguity.

From the reasoning presented in the proof of Proposition 8.11, we obtain the following corollary for any triangular **t**-module whose quotient ranks are greater than 1.

Corollary 8.12. *Let $\Upsilon = \Upsilon\left(\underline{\psi}, \underline{\delta}\right)$ be a triangular $\mathbf{t}$-module of dimension n, with no nilpotence and $\operatorname{rk}\psi_j > 1$ for $j = 1, 2, \ldots, n$. Then $\delta_i^\vee$ has the following bloc matrix form over $K(\Upsilon)$*

$$\delta_i^\vee = \Big[D_{i+1,i} \mid \cdots \mid D_{i+1,2} \mid D_{i+1,1} \Big], \tag{8.12}$$

where $D_{i+1,j}$ for $j = 1, 2 \ldots, i$ are matrices of size $r_i - 1 \times r_j - 1$, in which only the last column has nonzero entries.

9. Double dual t-module and Cartier-Nishi theorem

Classical Cartier–Nishi theorem asserts that one may recover an abelian variety from its dual via the Ext functor. In the category of **t**-modules an analogous result is known to hold for the duality of Drinfeld modules [PR03], as well as for strictly pure **t**-modules without nilpotence [KK26]. We will show that the same phenomenon occurs for triangular **t**-modules allowing for LD-biderivations.

9.1. **Double dual module.** In the first step, we show the double dual analogue of Theorem 8.2. We note that the double dual module $\left(\Upsilon^\vee\right)^\vee$ will be denoted as $\Upsilon^{\vee\vee}$, as this notation allows a more concise presentation.

Theorem 9.1. *Let $\Upsilon = \Upsilon\left(\underline{\psi}, \underline{\delta}\right)$ be a triangular $\mathbf{t}$-module with no nilpotence such that $r_i := \operatorname{rk}\psi_i > 1$ for $i = 1, \ldots, n$.*
Then

- *(i) double dual $\Upsilon^{\vee\vee}$ is a $\mathbf{t}$-module given by the $\tau-$composition series of the following form:*

$$\Upsilon^{\vee\vee}_1 \hookrightarrow \Upsilon^{\vee\vee}_2 \hookrightarrow \cdots \hookrightarrow \Upsilon^{\vee\vee}_n = \Upsilon^{\vee\vee},$$

- *(ii) there exists the following exact sequence of $\mathbf{t}$-modules*

$$0 \longrightarrow \Upsilon^{\vee\vee} \longrightarrow \operatorname{Ext}^1_\tau(\Upsilon^\vee, C) \longrightarrow \mathbb{G}_a^s \longrightarrow 0$$

  where $s = \sum_{l=1}^{n} \operatorname{rk}\psi_l - n$
- *(iii) Every map $f : \Upsilon \longrightarrow \underline{\Upsilon}$, of $\mathbf{t}$-modules that satisfy the assumptions of this theorem, induces a map $f^{\vee\vee} : \Upsilon^{\vee\vee} \longrightarrow \underline{\Upsilon}^{\vee\vee}$.*

Proof. The proof of this theorem follows the lines of the proof of Theorem 8.2. We leave details to the reader. $\square$

Remark 9.1. Our goal is to obtain the Cartier–Nishi theorem in a form analogous to that obtained for Drinfeld modules [PR03, Theorem 1.1], and also for strictly pure $\mathbf{t}$-modules [KK26, Theorem 1.2]. By Theorem 9.1, it suffices to show that $\Upsilon^{\vee\vee} \cong \Upsilon$.

Remark 9.2. From the previous theorem it follows that the $\mathbf{t}$-module $\Upsilon^{\vee\vee}$ has the following matrix form:

$$\Upsilon_t^{\vee\vee} = \begin{pmatrix} \psi_n^{\vee\vee} & 0 & \cdots & 0 & 0 & 0 \\ & (\psi_{n-1})^{\vee\vee} & \cdots & 0 & 0 & 0 \\ & & \ddots & \vdots & \vdots & \vdots \\ (\delta_{n-1})^{\vee\vee} & (\delta_{n-2})^{\vee\vee} & & \psi_3^{\vee\vee} & 0 & 0 \\ & & \cdots & \delta_2^{\vee\vee} & \psi_2^{\vee\vee} & 0 \\ & & & & \delta_1^{\vee\vee} & \psi_1^{\vee\vee} \end{pmatrix},$$

Recall that from the proof of [KK26, Theorem 3.4], it follows that

$$\psi_l^{\vee\vee} = \frac{1}{a_{l,r_l}} \cdot \psi_l \cdot a_{l,r_l} \quad \text{for} \quad l = 1, 2, \dots, n.$$

Thus the module $\Upsilon^{\vee\vee}$ has sub-quotients of the form $\dfrac{1}{a_{l,r_l}} \cdot \psi_l \cdot a_{l,r_l}$. In order to return to the initial sub-quotients ψ_l, we will consider the conjugate of $\Upsilon^{\vee\vee}$ by the diagonal matrix $D = \operatorname{diag}\left(a_{n,r_n}, \cdots, a_{1,r_1}\right)$, where a_{j,r_j} is the leading term of ψ_j:

$${}^{D}\Upsilon^{\vee\vee} := \operatorname{diag}\left(a_{n,r_n}, \cdots, a_{1,r_1}\right) \cdot \Upsilon^{\vee\vee} \cdot \operatorname{diag}\left(a_{n,r_n}, \cdots, a_{1,r_1}\right)^{-1}.$$

Then the biderivations for ${}^{D}\Upsilon^{\vee\vee}$ are given by the following formulas:

$${}^{D}\delta_l^{\vee\vee} = \operatorname{diag}\left(a_{l,r_l}, \cdots, a_{1,r_1}\right) \cdot \delta_l^{\vee\vee} \cdot \frac{1}{a_{l+1,r_{l+1}}} \quad \text{for} \quad l = 1, 2, \dots, n-1.$$

In the case of t-modules allowing for LD-biderivations we will show that

$$\delta_l = {}^{D}\delta_l^{\vee\vee}$$

from which it follows immediately that $\Upsilon \cong \Upsilon^{\vee\vee}$.

In order to prove that the Cartier-Nishi theorem holds for triangular $\mathbf{t}$-modules allowing for LD-biderivations, we first need to determine the precise form of the double dual module for this class of $\mathbf{t}$-modules. We begin with the following lemma:

Lemma 9.2. *Let $\Upsilon = \left(\underline{\psi}, \underline{\delta}\right)$ be a triangular* $\mathbf{t}$*-module with no nilpotence, such that $r_l := \operatorname{rk}\psi_l > 1$ for $l = 1, 2, \ldots, n$. Then*

$$(i) \quad \operatorname{Ext}^1_\tau(\Upsilon^\vee, C) \cong \Big[\underbrace{K\{\tau\}_{<1}, \ldots, K\{\tau\}_{<1}, K\{\tau\}_{<2}}_{\operatorname{rk}\psi_n - 1-\text{terms}} \Big| \cdots$$

$$\cdots \Big| \underbrace{K\{\tau\}_{<1}, \ldots, K\{\tau\}_{<1}, K\{\tau\}_{<2}}_{\operatorname{rk}\psi_2 - 1-\text{terms}} \Big| \underbrace{K\{\tau\}_{<1}, \ldots, K\{\tau\}_{<1}, K\{\tau\}_{<2}}_{\operatorname{rk}\psi_1 - 1-\text{terms}} \Big]$$

$$(ii) \quad \Upsilon^{\vee\vee} \cong \Big[\underbrace{0, 0, \ldots, 0, K\tau}_{\operatorname{rk}\psi_n - 1-\text{terms}} \Big| \cdots \Big| \underbrace{0, 0, \ldots, 0, K\tau}_{\operatorname{rk}\psi_2 - 1-\text{terms}} \Big| \underbrace{0, 0, \ldots, 0, K\tau}_{\operatorname{rk}\psi_1 - 1-\text{terms}} \Big]$$

Proof. Recall that $\operatorname{Ext}^1_\tau(\Upsilon^\vee, C) \cong \operatorname{Der}(\Upsilon^\vee, C)/\operatorname{Der}_{in}(\Upsilon^\vee, C)$, where $\operatorname{Der}(\Upsilon^\vee, C) \cong \prod\limits_{l=1}^{n} \operatorname{Mat}_{1\times \operatorname{rk}\psi_l - 1}\big(K\{\tau\}\big)$. Every element of $\operatorname{Der}(\Upsilon^\vee, C)$ will be identified with the matrix consisting of l blocs. We note that in this proof, and in the remainder of the paper, blocs will always be numbered from right to left, whereas the elements within each block will be indexed in the usual manner from left to right. Denote by $E_{i|l}$ the matrix belonging to $\operatorname{Mat}_{1\times \operatorname{rk}\psi_l - 1}(K)$, which as the $i-$th entry (counted from left to right) has 1 and has zeros at all other places. Additionally, by $\big[\big|E_{i|l}\big|\big]$ we denote the bloc matrix from $\prod\limits_{l=1}^{n} \operatorname{Mat}_{1\times \operatorname{rk}\psi_l - 1}\big(K\{\tau\}\big)$, where the $l-$th bloc from the right is equal to $E_{i|l}$ and all the other blocs are equal to zero. Consider the following generators of the space $\operatorname{Der}_{in}(\Upsilon^\vee, C)$ for $\mu \in K$ and $k = 0, 1, 2, \ldots$

(9.1)

$$\begin{aligned}\delta^{\left([|E_{i|l}|]\mu\tau^k\right)} &= \mu\tau^k \big[\big|E_{i|l}\big|\big]\Upsilon^\vee - C\mu\tau^k\big[\big|E_{i|l}\big|\big] = \\ &= \Big[\underbrace{\mathbf{0} \,\Big|\, \cdots \,\Big|\, \mathbf{0}}_{(n-l)-\text{blocks}} \Big| \delta^{(E_{i|l}\mu\tau^k)}_{\psi_l^\vee, C} \Big| E_{i|l}\cdot\mu\tau^k D_{l,l-1} \Big| \cdots \Big| E_{i|l}\cdot\mu\tau^k D_{l,1}\Big],\end{aligned}$$

where $\delta^{(E_{i|l}\mu\tau^k)}_{\psi_l^\vee, C} \in \operatorname{Der}_{in}(\psi_l^\vee, C)$ and $\delta^\vee_{l-1} = \Big[D_{l,l-1} \Big| \ldots \Big| D_{l,2} \Big| D_{l,1}\Big]$.

From Proposition 8.10 we get $\delta^{\left([|E_{i|l}|]\mu\tau^k\right)} \in \operatorname{Der}_0(\Upsilon^\vee, C)$ for every $k = 0, 1, 2, \ldots$. Thus (ii) follows from (i).

Notice that from Corollary 8.12 it follows that in every matrix

$$E_{i|l}\cdot\mu\tau^k D_{l,j} \quad \text{for} \quad j = l-1, l-2, \ldots, 2, 1$$

only the last column is nonzero.

Let $\psi_l = \theta + \sum\limits_{s=1}^{r_l} a_{l,s}\tau^s$. Then from the form of (8.3) it follows that the inner biderivation $\delta^{(E_{i|l}\mu\tau^k)}_{\psi_l^\vee, C}$ is one of two types. For $i = 1$ we obtain

$$\delta^{(E_{1|l}\cdot\mu\tau^k)}_{\psi_l^\vee, C} = \Big[\mu\cdot\big(\theta^{(k)} - \theta\big)\tau^k - \mu^{(1)}\tau^{k+1}, 0, \ldots, 0, -\frac{\mu\cdot a^{(k)}_{l,1}}{a^{(k)}_{l,r_l}}\tau^{k+1} + \frac{\mu}{a^{(k+1)}_{l,r_l}}\tau^{k+2}\Big]$$

whereas for $i = 2, \cdots, r_l - 1$ one gets

$$\delta^{(E_{i|l} \cdot \mu\tau^k)}_{\psi_l^\vee, C} = \Big[\ \underbrace{0, \dots, 0}_{(i-2)-\text{terms}}\ , \mu \cdot \tau^{k+1}, \mu \cdot \big(\theta^{(k)} - \theta\big)\tau^k - \mu^{(1)}\tau^{k+1}, 0, \dots, 0, -\frac{\mu \cdot a^{(k)}_{l,i}}{a^{(k)}_{l,r_l}}\tau^{k+1}\Big],$$

where for $i = r_l - 1$ the last term is of the following form:

$$\mu \cdot \big(\theta^{(k)} - \theta\big)\tau^k - \mu^{(1)}\tau^{k+1} - \frac{\mu \cdot a^{(k)}_{l,i}}{a^{(k)}_{r_l}}\tau^{k+1}$$

Thus, we have two types of biderivations $\delta^{\left(\left[\left|E_{i|l}\right|\right]\mu\tau^k\right)}$ for the chosen $l-$th bloc. The biderivation of the first type for $i = 1$, by means of which the last column (counted from the left) of the l-th bloc will be reduced. Then the biderivations of the second type for $i = 2, 3, \dots, \operatorname{rk}\psi_l - 1$, will be used for reduction of the $i-1$-th column in the $l-$th bloc. As a result of such a reduction at the i-th position, a term of the same degree as that just reduced will appear. Therefore the biderivations of the second type "shift" the highest term by one place to the right. Additionally, the new terms will appear only in the last columns of the blocks located to the right of the $l-$th bloc.

Now we can explain the reduction of any inner biderivation from $\mathrm{Der}(\Upsilon^\vee, C)$. During the reduction process the consecutive blocs from left to right will be changed. Assume that we have reduced all the blocs to the left of the $l-$th bloc. The reduction in the l-th bloc will be performed according to the following procedure:

(1) let M be the maximal degree of terms in the $l-$th bloc
(2) denote by m the index of the first from the left column with the term of degree M.
(3) if $M = 2$ and m is the index of the last column, then we have finished the reduction of the bloc.
(4) if m is the index of the last column, then we reduce this term by means of the biderivation of the first type $\delta^{\left(\left[\left|E_{1|l}\right|\right]\mu\tau^k\right)}$ for appropriate $\mu \in K$ and $k \in \mathbb{Z}_{\geq 0}$. Then we go to step (1).
(5) if m is not an index of the last column, then we reduce it by means of the biderivation of the second type of the form $\delta^{\left(\left[\left|E_{m+1|l}\right|\right]\mu\tau^k\right)}$ for appropriately chosen $\mu \in K$ and $k \in \mathbb{Z}_{\geq 0}$. As a result of this operation, a term of maximal degree is now in the $m+1-$st column. We return to step (3).

As a result of applying this procedure, we see that all terms in the $l-$th bloc have expected degrees. Since

$$\deg_\tau \delta_{\psi_l^\vee, C}^{(E_{1|l}\mu\tau^k)} = k+2, \quad \deg_\tau \delta_{\psi_l^\vee, C}^{(E_{i|l}\mu\tau^k)} = k+1, \quad \text{for} \quad i = 2,3,\cdots \operatorname{rk}\psi_l - 1,$$

we see that two different bloc matrices in

$$\Big[\underbrace{K\{\tau\}_{<1},\ldots,K\{\tau\}_{<1},K\{\tau\}_{<2}}_{\operatorname{rk}\psi_n - 1-\text{terms}} \;\Big|\; \cdots \Big|\; \underbrace{K\{\tau\}_{<1},\ldots,K\{\tau\}_{<1},K\{\tau\}_{<2}}_{\operatorname{rk}\psi_1 - 1-\text{terms}}\Big]$$

represent different elements in $\operatorname{Ext}^1_\tau(\Upsilon^\vee, C)$. $\square$

9.2. **The case of LD-biderivations.** We consider triangular $\mathbf{t}$-modules that allow for LD-biderivations.

Recall the procedure for determining $\psi^{\vee\vee}$ for a Drinfeld module of the form $\psi = \theta + \sum\limits_{s=1}^{r} a_s\tau^s$. Since $\psi^{\vee\vee} \cong \big[0,0,\cdots,0,K\tau\big]$, in order to determine the $\mathbf{t}$-module structure on $\psi^{\vee\vee}$, we have to find the result of the action $t * \big[0,0,\cdots,0,c\tau\big] = \big[0,0,\cdots,0,\theta\tau + c^{(1)}\tau^2\big]$. For this we reduce by means of the following inner biderivation:

$$\delta_{\psi^\vee, C}^{\left(V_\psi(\mu)\right)} = \left[0,\ldots,0,\frac{\mu}{a_r^{(1)}}\tau^2 - \sum_{s=1}^{r}\frac{a_s}{a_r}\mu^{(s-1)}\cdot\tau\right] \quad \text{where} \tag{9.2}$$

$$\mu = c^{(1)}a_r^{(1)} \quad \text{and} \quad V_{\psi_l}(\mu) := \big[\mu,\mu^{(1)},\cdots,\mu^{(r-2)}\big]\tau^0.$$

Then we write the result in the standard basis and obtain the form of $\psi^{\vee\vee}$. The inner biderivation $\delta_{\psi^\vee, C}^{\left(V_\psi(\mu)\right)}$ will be used further on to determine $\Upsilon^{\vee\vee}$ for a triangular $\mathbf{t}$-module Υ.

Theorem 9.3 (Cartier-Nishi Theorem)**.** *Let $\Upsilon = \big(\underline{\psi},\underline{\delta}\big)$ be a triangular $\mathbf{t}$-module with no nilpotence, such that $r_l := \operatorname{rk}\psi_l > 1$ for $l = 1,2,\ldots,n$. If Υ allows for LD-biderivations, then $\Upsilon^{\vee\vee} \cong \Upsilon$ over $K(\Upsilon)$, and there exists the following exact sequence of $\mathbf{t}$-modules over $K(\Upsilon)$*

$$0 \longrightarrow \Upsilon \longrightarrow \operatorname{Ext}^1_\tau(\Upsilon^\vee, C) \longrightarrow \mathbb{G}_a^s \longrightarrow 0$$

where $s = \sum\limits_{l=1}^{n} \operatorname{rk}\psi_l - n$

Proof. Since Υ admits LD-biderivations, after changing the field to $K(\Upsilon)$, we may assume that Υ satisfies condition (4.6).

The existence of this exact sequence will follow from the isomorphism $\Upsilon^{\vee\vee} \cong \Upsilon$ applied to Theorem 9.1. So we will focus on establishing this isomorphism. According to Remark 9.2, it is enough to show that

$${}^{D}\delta_l^{\vee\vee} = \delta_l \quad \text{for} \quad l = 1,2,\ldots,n-1.$$

Let $\psi_l = \theta + \sum_{s=1}^{r_l} a_{l,s}\tau^s$. We will induct on the number of biderivations.

Let $\Upsilon_2 = \left[\begin{array}{c|c} \psi_2 & 0 \\ \hline \delta_1 & \psi_1 \end{array}\right]$. From Corollary 8.8 it follows that the correspondence $\delta_1 \longmapsto \delta_1^{\vee\vee}$ is $\mathbb{F}_q-$linear. Therefore it is enough to find the double dual biderivation for generators of the form $d\tau^h$ for $1 \le h < r_2$ and to show that

$$a_{1,r_1} \cdot \left(d\tau^h\right)^{\vee\vee} \cdot \frac{1}{a_{2,r_2}} = d\tau^h.$$

Recall that

$$\left(d\tau^h\right)^{\vee} = -\frac{d}{a_{1,r_1}} E_{h\times r_1 - 1}\tau,$$

From Lemma 9.2 it follows that

$$\Upsilon_2^{\vee\vee} \cong \Big[\underbrace{0, 0, \ldots, 0, K\tau}_{r_2-1-\text{terms}} \ \Big|\ \underbrace{0, 0, \ldots, 0, K\tau}_{r_1-1-\text{terms}}\Big].$$

Recall that $E_{i|l}$ denotes the matrix of dimensions $1 \times r_l - 1$, which at the $i-$th entry has value 1 (counted from left to right) and in addition zeros at all other entries, and

$$\big[\big|E_{i|2}\big|\big] = \Big[\ E_{i|2}\ \Big|\ \mathbf{0}\ \Big], \quad \big[\big|E_{i|1}\big|\big] = \Big[\ \mathbf{0}\ \Big|\ E_{i|1}\ \Big].$$

In order to determine the $F_q[t]-$module structure on $\Upsilon_2^{\vee\vee}$, one has to find an action of t on the generator $c\cdot\tau\big[\big|E_{r_2-1|2}\big|\big]$ written in the following basis:

$$\tau \cdot \big[\big|E_{r_2-1|2}\big|\big] = \Big[\ 0, \ldots, 0, \tau\ \Big|\ \mathbf{0}\ \Big], \qquad \tau \cdot \big[\big|E_{r_1-1|1}\big|\big] = \Big[\ \mathbf{0}\ \Big|\ 0, \ldots, 0, \tau\ \Big].$$

Then

$$t * c\tau \cdot \big[\big|E_{r_2-1|2}\big|\big] = C_t \cdot c\tau \cdot \big[\big|E_{r_2-1|2}\big|\big] = \Big[\ 0, \ldots, 0, \theta c\tau + c^{(1)}\tau^2\ \Big|\ \mathbf{0}\Big].$$

The summand $c^{(1)}\tau^2 \cdot \big[\big|E_{r_2-1|2}\big|\big]$ is reduced by means of the following inner biderivation given by equation (9.2):

$$\delta_{\Upsilon_2^{\vee},C}^{\left(\left[\left|V_{\psi_2}(\mu)\right|\right]\right)} = \left[\delta_{\psi_2^{\vee},C}^{\left(V_{\psi_2}(\mu)\right)}\Big|\ 0, \ldots, 0, -\frac{\mu^{(h-1)}d}{a_{1,r_1}}\tau\right]$$

where $\big[\big|V_{\psi_2}(\mu)\big|\big] := \big[V_{\psi_2}(\mu)\big|\ 0, \cdots, 0\ \big]$ for $\mu = c^{(1)} \cdot a_{2,r_2}^{(1)}$.

We obtain the following equality:

$$t * c\tau \cdot \big[\big|E_{r_2-1|2}\big|\big] = \left[\ \text{terms for } \psi_2^{\vee\vee}\ \Big|\ 0, \ldots, 0, \frac{\mu^{(h-1)}d}{a_{1,r_1}}\tau\right].$$

Substituting $\mu = c^{(1)}a^{(1)}_{r_2,2}$ and expressing $t * c\tau \cdot \big[\big|E_{r_2-1|2}\big|\big]$ in the basis one gets

$$t * c\tau \cdot \big[\big|E_{r_2-1|2}\big|\big] = \left[\ \psi_2^{\vee\vee}\ ,\ \frac{a^{(h)}_{2,r_2}d}{a_{1,r_1}}\tau^h\right]\Bigg|_{\tau=c}.$$

Hence $\big(d\tau^h\big)^{\vee\vee} = \dfrac{a^{(h)}_{2,r_2}d}{a_{1,r_1}}\tau^h = \dfrac{d}{a_{1,r_1}}\tau^h \cdot a_{2,r_2}$ and $a_{1,r_1} \cdot \big(d\tau^h\big)^{\vee\vee} \cdot \dfrac{1}{a_{2,r_2}} = d\tau^h.$

Assume that for a triangular $\mathbf{t}$-module of dimension $n-1$ we have

$${}^{D}\delta_l^{\vee\vee} = \delta_l \quad \text{for} \quad l = 1, 2, \dots, n-2.$$

and every double dual biderivation $\delta_l^{\vee\vee}$ came from determining the action $t * c\tau \cdot \big[\big|E_{r_l-1|l}\big|\big]$, where the following inner biderivation

$$\delta^{\left(\left[\left|V_{\psi_l}\right|\right](\mu)\right)}, \quad \text{for} \quad \mu = c^{(1)} \cdot a^{(1)}_{l,r_l}$$

was used in the reduction process.

Here, $\big[\big|V_{\psi_l}(\mu)\big|\big]$ is a bloc matrix where the only nonzero bloc is the $l-$th, counted from right to left, and is equal to $V_{\psi_l}(\mu)$.

Let Υ be a triangular $\mathbf{t}$-module of dimension n. By the inductive hypothesis

$${}^{D}\delta_l^{\vee\vee} = \delta_l \quad \text{for} \quad l = 1, 2, \cdots n-2.$$

Since the assignment $\delta_{n-1} \longmapsto (\delta_{n-1})^{\vee\vee}$ is $\mathbb{F}_q-$linear, it is enough to consider the case where $\delta_{n-1} = \Big[d_{n-1}\tau^{h_{n-1}}, \cdots, d_2\tau^{h_2}, d_1\tau^{h_1}\Big]^T$. Recall that by (8.9) we have

$$\delta_{n-1}^{\vee} = -\sum_{j=1}^{n-1} \frac{d_i}{a_{j,r_j}} E_{h_i \times r_j - 1}\tau.$$

From Lemma 9.2 it follows that

$$\Upsilon^{\vee\vee} \cong \Big[\underbrace{0, 0, \dots, 0, K\tau}_{r_n-1-\text{terms}}\ \Big|\ \cdots\ \Big|\ \underbrace{0, 0, \dots, 0, K\tau}_{r_2-1-\text{terms}}\ \Big|\ \underbrace{0, 0, \dots, 0, K\tau}_{r_1-1-\text{terms}}\Big].$$

Recall that $E_{i|l}$ denotes the matrix of dimensions $1 \times r_l - 1$, which at the $i-$th entry has value 1, zeros at all other entries, and $\big[\big|E_{i|l}\big|\big]$ denotes the bloc matrix where the $l-$th bloc (counted from the right) equals $E_{i|l}$ and all the other blocs are zero. In order to determine the $F_q[t]-$module structure on $\Upsilon^{\vee\vee}$, one has to find an action of t on the generators $c \cdot \tau \big[\big|E_{r_n|n}\big|\big]$ written in the following basis:

(9.3) $$\tau \cdot \big[\big|E_{r_n|n}\big|\big],\ \tau \cdot \big[\big|E_{r_{n-1}|n-1}\big|\big],\ \cdots,\ \tau \cdot \big[\big|E_{r_2|2}\big|\big],\ \tau \cdot \big[\big|E_{r_1|1}\big|\big].$$

Then

$$t * c\tau \cdot \big[\big|E_{r_n-1|n}\big|\big] = (\theta + \tau)c\tau \cdot \big[\big|E_{r_n-1|n}\big|\big] = \Big(\theta c\tau + c^{(1)}\tau^2\Big) \cdot \big[\big|E_{r_n-1|n}\big|\big]$$

The summand $c^{(1)}\tau^2 \cdot \big[\big|E_{r_n-1|n}\big|\big]$ is reduced by means of the following inner biderivation given by equation (9.2):

$$\delta\Big(\big[\big|V_{\psi_n}(\mu)\big|\big]\tau^0\Big) = \Bigg[\delta_{\psi_n,C}^{\left(V_{\psi_n}(\mu)\right)} \mid 0,\dots,0,-\frac{\mu^{(h_{n-1}-1)}d_{n-1}}{a_{n-1,r_{n-1}}}\tau \mid \cdots$$
$$\cdots \mid 0,\dots,0,-\frac{\mu^{(h_2-1)}d_2}{a_{2,r_2}}\tau \mid 0,\dots,0,-\frac{\mu^{(h_1-1)}d_1}{a_{1,r_1}}\tau \Bigg]$$

for $\mu = c^{(1)}\cdot a_{n,r_n}^{(1)}$. We obtain

$$t * c\tau \cdot \big[\big|E_{r_n-1|n}\big|\big] = \Bigg[\text{ terms for } \psi_2^{\vee\vee} \mid 0,\dots,0,\frac{\mu^{(h_{n-1}-1)}d_{n-1}}{a_{n-1,r_{n-1}}}\tau \mid \cdots$$
$$\cdots \mid 0,\dots,0,\frac{\mu^{(h_2-1)}d_2}{a_{2,r_2}}\tau \mid 0,\dots,0,\frac{\mu^{(h_1-1)}d_1}{a_{1,r_1}}\tau \Bigg]$$

Substituting $\mu = c^{(1)}a_{r_n,n}^{(1)}$ and expressing $t*c\tau\cdot\big[\big|E_{r_n-1|n}\big|\big]$ in the basis (9.3) one gets

$$t * c\tau \cdot \big[\big|E_{r_n-1|n}\big|\big] = \Bigg[\text{ terms for } \psi_2^{\vee\vee},\ \frac{a_{r_n,n}^{(h_{n-1})}d_{n-1}}{a_{n-1,r_{n-1}}}\tau^{h_{n-1}},\cdots,$$
$$\frac{\frac{a_{r_n,n}^{(h_2)}d_2}{a_{2,r_2}}\tau^{h_2},\ a_{r_n,n}^{(h_1)}d_1}{a_{1,r_1}}\tau^{h_1}\Bigg]\Bigg|_{\tau=c}.$$

Then

$$(\delta_{l-1})^{\vee\vee} = \left[\frac{a_{r_n,n}^{(h_{n-1})}d_{n-1}}{a_{n-1,r_{n-1}}}\tau^{h_{n-1}},\cdots,\frac{a_{r_n,n}^{(h_2)}d_2}{a_{2,r_2}}\tau^{h_2},\ \frac{a_{r_n,n}^{(h_1)}d_1}{a_{1,r_1}}\tau^{h_1}\right]^T.$$

So,

$${}^{D}(\delta_{n-1})^{\vee\vee} = \operatorname{diag}\left(a_{n-1,r_{n-1}},\cdots,a_{1,r_1}\right)\cdot(\delta_{n-1})^{\vee\vee}\cdot\frac{1}{a_{n,r_n}} = \delta_{n-1}.$$

□

As an immediate consequence of Theorem 9.3 and Corollary 4.3, we obtain the following fact.

Corollary 9.4. *Let $\Upsilon = \left(\underline{\psi},\underline{\delta}\right)$ be a triangular $\mathbf{t}$-module with no nilpotence, such that*

$$1 < \operatorname{rk}\psi_1 < \operatorname{rk}\psi_2 < \cdots < \operatorname{rk}\psi_n.$$

Then $\Upsilon^{\vee\vee} \cong \Upsilon$ over $K(\Upsilon) = K$, and there exists the following exact sequence of $\mathbf{t}$-modules over K

$$0 \longrightarrow \Upsilon \longrightarrow \operatorname{Ext}^1_\tau(\Upsilon^\vee, C) \longrightarrow \mathbb{G}_a^s \longrightarrow 0$$

where $s = \sum_{l=1}^{n} \operatorname{rk} \psi_l - n$ □

Similarly, from Corollary 4.4 and Theorem 9.3 we obtain the following corollary.

Corollary 9.5. *Let* $\Upsilon = \left(\underline{\psi}, \underline{\delta}\right)$ *be a triangular* $\mathbf{t}$*-module, defined over* K*, with no nilpotence, such that*

$$1 < \operatorname{rk} \psi_1 \leq \operatorname{rk} \psi_2 \leq \cdots \leq \operatorname{rk} \psi_n.$$

Then $\Upsilon^{\vee\vee} \cong \Upsilon$ *over* $K(\Upsilon)$ *and there exists the following exact sequence of* $\mathbf{t}$*-modules over* $K(\Upsilon)$

$$0 \longrightarrow \Upsilon \longrightarrow \operatorname{Ext}^1_\tau(\Upsilon^\vee, C) \longrightarrow \mathbb{G}_a^s \longrightarrow 0$$

where $s = \sum_{l=1}^{n} \operatorname{rk} \psi_l - n.$ □

Remark 9.3. Notice that in order to define the dual t-module $\Phi^\vee$ using the methods of Taguchi, the assumption that Φ is strictly pure is essential. Indeed, in [KK26] it was shown that for a strictly pure t-module Φ without nilpotence, the Cartier–Taguchi dual module coincides with that defined as $\operatorname{Ext}^1_{\tau,0}(\Phi, C)$. However, triangular t-modules are in general not strictly pure (cf. Proposition 4.5). This indicates that the construction of the dual module via $\operatorname{Ext}^1_{\tau,0}(\Phi, C)$ is in fact more general.

The following example shows that in Theorem 9.3 the nilpotence condition is necessary.

Example 9.1. Consider the $\mathbf{t}$-module

$$\Upsilon = \left(\begin{array}{c|c} \theta + \tau^3 & 0 \\ \hline 1 & \theta + \tau^2 \end{array}\right).$$

Then

$$\Upsilon^\vee = \left(\begin{array}{ccc|c} \theta & \tau^2 & 0 & \tau \\ \tau & \theta & 0 & 0 \\ 0 & 0 & \theta & \left(\theta - \theta^{(1)}\right)\tau \\ \hline 0 & 0 & \tau & \theta + \tau^2 \end{array}\right).$$

For determination of $\left(\Upsilon^\vee\right)^\vee$ we have to assume that K is a perfect field. Then from the reduction algorithm we obtain that

$$\left(\Upsilon^\vee\right)^\vee = \begin{pmatrix} \theta + \tau^3 & 0 & 0 \\ \tau & \theta & \left(\theta^{(1)} - \theta\right) + \tau^2 \\ 0 & \tau & \theta \end{pmatrix} \not\cong \Upsilon.$$

Our results show that the Cartier–Nishi theorem holds for triangular $\mathbf{t}$-modules allowing for LD-biderivations, and we have provided an explicit sufficient condition ensuring that a triangular $\mathbf{t}$-module allows for LD- biderivations. At present, it is natural to ask to what extent this hypothesis is actually necessary. All triangular $\mathbf{t}$-modules we have tested outside those allowing for LD-biderivations still satisfy the Cartier–Nishi formula, and no counterexample is currently known. This leads to the following question:

Question 9.6. *For which triangular* $\mathbf{t}$*-modules does the Cartier–Nishi theorem hold? Does it hold for all triangular* $\mathbf{t}$*-modules, or does there exist a triangular* $\mathbf{t}$*-module for which the theorem does not hold?*

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Institute of Mathematics, Department of Exact and Natural Sciences, University of Szczecin, ul. Wielkopolska 15, 70-451 Szczecin, Poland

Email address: dawidedmundkedzierski@gmail.com

Institute of Mathematics, Department of Exact and Natural Sciences, University of Szczecin, ul. Wielkopolska 15, 70-451 Szczecin, Poland

Email address: piotrkras26@gmail.com